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Off-mass-shell massless particles and the Weyl group in light-cone coordinates

D J Almond
Department of Physics, Queen Mary College, Mile End Road, London E1 4NS, UK

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Abstract. We extend our previous work on the description of virtual (off-mass-shell) relativistic particles by unitary irreducible representations of the Weyl group (the group of Poincaré transformations and dilatations on Minkowski space–time) to the case of particles with zero on-mass-shell mass. We show that the structure of the d-dimensional Weyl group Lie algebra on the light-cone is that of the group of inhomogeneous Galilei transformations and non-relativistic dilatations on a Euclidean space plus time of $(d-2)$ space dimensions. The 'dilatation' generators of the two non-relativistic algebras are $D\pm M^{0(d-1)}$, and an infinite transformation of these operators takes the momentum operator P^μ into $(P^\pm/\sqrt{2}, \mathbf{0}, \pm P^\pm/\sqrt{2})$ respectively, where $P^\pm=(P^0\pm P^{(d-1)})/\sqrt{2}$. The transformations therefore take a state of an off-mass-shell massless particle, of arbitrary momentum, onto the mass-shell. We work out how the spin and position operators transform under the infinite transformations, and express the Weyl group generators in terms of the transformed operators. We also construct a canonical form for the transformed operators and generators, and discuss the transformation properties of the single-particle states under a unitary operator of the Weyl group. The case $d=4$ is discussed in detail and, in particular, we extend Weinberg's theorem on massless irreducible representations of the $d=4$ Poincaré group to the case of the Weyl group. In an appendix, we treat the position–spin–momentum algebra of the Weyl group classically (i.e. as Poisson brackets) and find the Dirac brackets compatible with the constraints $P^2\approx 0$, $R^+-\beta\approx 0$.

1. Introduction

We have shown (Almond 1973a, § III) that off-mass-shell relativistic particles are described by unitary irreducible representations of the Weyl group, the group of Poincaré transformations and dilatations acting on Minkowski space–time. The original work was for space–time of dimension $d=4$, but it was later shown that the generalisation to arbitrary d was straightforward (Almond 1981a). However, these papers dealt only with particles of non-zero on-mass-shell mass ('massive particles'), and the case of particles with zero on-mass-shell mass ('massless particles') remained unresolved†, the problem being essentially that a dilatation transforms the momentum operator P^μ into $e^{-\alpha}P^\mu$, and so a state of an off-mass-shell massless particle with time-like or space-like momentum can never be put onto the mass-shell (i.e. transformed into a state with light-like momentum) by a dilatation transformation. In this paper we show that the correct framework for treating massless particles is the Weyl group Lie algebra in light-cone coordinates. The operators $D\pm M^{0(d-1)}$ (where D and

† The statement in the 'note added in proof' of Almond (1973a), that massless particles are described by the zero-mass limit of massive particles, is trivial and should be ignored.

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$M^{0(d-1)}$ are the Hilbert space generators of dilatations and boosts in the $(d-1)$ direction, respectively) occur naturally in the light-cone algebra, and an infinite transformation generated by these operators takes a state of an off-mass-shell massless (or for that matter, massive) particle with time-like or space-like momentum into one with light-like momentum.

The layout of the paper is as follows. In § 2 we analyse the Lie algebra of the d-dimensional Weyl group in light-cone coordinates. It is well known that the light-cone structure of the $d=4$ Poincaré group Lie algebra (Bacry and Chang 1968, Bardakci and Halpern 1968, Susskind 1968, Kogut and Soper 1970, Bjorken *et al* 1971, Biedenharn *et al* 1973, Kogut and Susskind 1973, Staunton 1973, Biedenharn and van Dam 1974) is just that of the extended Galilei group Lie algebra (Lévy-Leblond 1963) in two space dimensions. This result was generalised to arbitrary d by Hu (1972), who showed that the light-cone structure of the Poincaré group Lie algebra in this case is that of the extended Galilei group Lie algebra in $(d-2)$ space dimensions. It has also been shown (Del Guidice *et al* 1972, Domokos 1972, Burdet *et al* 1973) that the light-cone structure of the $d=4$ conformal group Lie algebra is that of the Schrödinger group (i.e. the group of inhomogeneous Galilei transformations, non-relativistic dilatations $(t', \boldsymbol{x}')=(\lambda^2 t, \lambda\boldsymbol{x})$ and 'expansions' $(t', \boldsymbol{x}')=(t/(1+\alpha t), \boldsymbol{x}/(1+\alpha t))$ (Burdet and Perrin 1972, Hagen 1972, Niederer 1972, Roman *et al* 1972)) in two space dimensions. With these results in mind, it is not surprising that we find the light-cone structure of the Lie algebra of the d-dimensional Weyl group to be that of the group of inhomogeneous Galilei transformations and non-relativistic dilatations (Almond 1973a, § II, Bez 1976) in $(d-2)$ space dimensions. We work out the 'spin', 'time' and 'position' operators of the two 'non-relativistic' algebras, and their transformation properties under parity and time-reversal.

In § 3 we show that an infinite transformation generated by the 'dilatation' generators of the two 'non-relativistic' algebras, $D \pm M^{0(d-1)}$, takes the momentum operator P^μ into $(P^\pm/\sqrt{2}, \mathbf{0}, \pm P^\pm/\sqrt{2})$, and so takes a state of an off-mass-shell massless particle onto the mass-shell. In § 3.1 we work out how the different components of the relativistic spin and position operators transform under an infinite transformation generated by $D+M^{0(d-1)}$, and find that some of the components go over into the 'spin', 'time' and 'position' operators of the corresponding 'non-relativistic' algebra. We work out the algebra satisfied by these transformed operators, and give a canonical representation of them. We also express the Weyl group generators in terms of the transformed operators, and therefore have a canonical representation of the generators too. In § 3.2 we evaluate the effect of an infinite transformation generated by $D+M^{0(d-1)}$ on a state of an off-mass-shell massless particle with time-like momentum. Because of the complicated nature of the spin for general d, we restrict ourselves to a particle moving in the $(d-1)$ direction. The on-mass-shell states are in a different Hilbert space to the off-mass-shell states, since the infinite transformation is non-unitary (i.e. it has no inverse). We work out the transformation properties of a general on-mass-shell state under a unitary operator of the Weyl group. Section 3.3 is devoted to a study of the effect of the other infinite transformation generated by $D-M^{0(d-1)}$ on the operators and states. Section 3.4 deals with how the parity and time-reversal operators transform the two Hilbert spaces of on-mass-shell states into each other.

Section 4 is devoted to a study of the case $d=4$. In § 4.1 we work out how the Pauli–Lubanski spin pseudovector and the helicity operator transform under the two light-cone transformations. We also give a canonical form for the Weyl group generators and show that for $p^2=0$ it agrees with the expressions obtained for the Poincaré group generators by previous authors (Lomont and Moses 1962, Chakrabarti 1966). In

§ 4.2 we evaluate the effect of the two light-cone transformations on a state of an off-mass-shell massless particle with time-like four-momentum, and show how the states in the two on-mass-shell Hilbert spaces are related by parity and time-reversal. In § 4.3 we extend Weinberg's theorem (Weinberg 1964a, Nieder and O'Raifeartaigh 1974, ch VII) on the description of massless irreducible representations of the Poincaré group by irreducible representations of the homogeneous Lorentz group. In addition to Weinberg's result that an on-mass-shell massless particle of helicity $\lambda > 0$ can only be described by irreducible representations of the Lorentz group $(k, k+\lambda)$ (with k any integer or half-integer such that $k \geqslant 0$, $k+\lambda \geqslant 0$), we find that the different helicity components of an off-mass-shell massless particle taken onto the mass-shell must all have the same value of $2k+\lambda$. A corresponding result holds for $\lambda < 0$.

Section 5 is a Conclusion. Appendix 1 discusses the extended Lie algebra of the group of inhomogeneous Galilei transformations and non-relativistic dilatations in an arbitrary number of space dimensions. Appendix 2 is concerned with the application of the second-class constraints $\varphi_1 \equiv P^2 \approx 0$, $\varphi_2 \equiv R^+ - \beta \approx 0$ to the d-dimensional Weyl group Lie algebra in classical (Poisson bracket) form. Here a novel feature appears: since the Poisson brackets $\{R^\mu, R^+ - \beta\}$ and $\{W^{\mu\nu}, R^+ - \beta\}$ are $\sim 1/P^2$, we find that, for components of R^μ other than R^+, the redefined quantities $R^\mu - \{R^\mu, \varphi_\alpha\} C^{-1}_{\alpha\beta} \varphi_\beta$ (with $C_{\alpha\beta} = \{\varphi_\alpha, \varphi_\beta\}$), which should have vanishing Poisson brackets with the constraints, are not weakly equal to R^μ; similarly for the components of $W^{\mu\nu}$. However, we find that the linear combinations of R^μ and $W^{\mu\nu}$ which *are* weakly equal to their redefined expressions are just those which arose quantum mechanically in § 3.1 as the light-cone transforms of the spin and position operators. Appendix 3 contains a proof of an operator identity, and appendix 4 deals with some properties of the finite-dimensional irreducible representations of $\mathrm{SO}(d-1, 1)$.

2. The d-dimensional Weyl group in light-cone coordinates

The Weyl group is a $[\frac{1}{2}d(d+1)+1]$-parameter group consisting of the homogeneous Lorentz transformations, displacements and dilatations acting on Minkowski space–time $x^\mu = (x^0, x^1, \ldots, x^{d-1}) = (x^0, x^i)$ according to

$$x'^\mu = \lambda L^\mu{}_\nu x^\nu + a^\mu. \tag{2.1}$$

Here λ, the dilatation, is a real positive constant, a^μ is a real constant vector displacement, and $L^\mu{}_\nu$, the homogeneous Lorentz transformation, is a matrix satisfying

$$L^\mu{}_\nu g_{\mu\rho} L^\rho{}_\sigma = g_{\nu\sigma}, \tag{2.2}$$

where the Minkowski metric tensor, $g_{\mu\nu}$, is given by

$$g_{00} = 1, \qquad g_{ij} = -\delta_{ij}, \qquad g_{0i} = 0 = g_{i0}. \tag{2.3}$$

Due to the presence of an arbitrary phase factor in quantum mechanics, physically we are interested in the direct sum of the Weyl Lie algebra with that of U(1), i.e.†

$$\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}(M^{\mu\rho} g^{\nu\sigma} - M^{\mu\sigma} g^{\nu\rho} + M^{\nu\sigma} g^{\mu\rho} - M^{\nu\rho} g^{\mu\sigma}), \\
&[M^{\mu\nu}, P^\sigma] = -\mathrm{i}(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}), \\
&[P^\mu, P^\nu] = 0, \qquad [D, M^{\mu\nu}] = 0, \qquad [D, P^\mu] = -\mathrm{i}P^\mu, \\
&[M^{\mu\nu}, M] = 0, \qquad [P^\mu, M] = 0, \qquad [D, M] = 0,
\end{aligned} \tag{2.4}$$

† Throughout the paper, we put $\hbar$ equal to unity.

where $M^{\mu\nu}$, P^μ, D and M are the Hermitian operators generating Lorentz transformations, translations, dilatations and U(1) transformations, respectively, in Hilbert space. The algebra, equations (2.4), has been discussed in detail for $d=4$ (Almond 1973a, § III, 1974) and for general d (Almond 1981a). To see its physical significance, we define the position operator

$$R^\mu = \frac{1}{2}\left[\frac{P^\mu}{P^2}, D\right]_+ - \frac{[P_\nu, M^{\mu\nu}]_+}{2P^2}, \tag{2.5}$$

where $[A, B]_+ \equiv AB + BA$, and the spin operator (Nyborg 1964, Kolsrud 1967)

$$W^{\mu\nu} = \left(g^{\mu\rho} - \frac{P^\mu P^\rho}{P^2}\right)\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) M_{\rho\sigma}, \tag{2.6}$$

which satisfy

$$\begin{aligned}
&[M^{\mu\nu}, R^\sigma] = -\mathrm{i}(R^\mu g^{\nu\sigma} - R^\nu g^{\mu\sigma}),\\
&[R^\mu, P^\nu] = -\mathrm{i}g^{\mu\nu}, \qquad [D, R^\mu] = \mathrm{i}R^\mu, \qquad [M, R^\mu] = 0,\\
&[R^\mu, R^\nu] = -\mathrm{i}W^{\mu\nu}/P^2,\\
&[M^{\mu\nu}, W^{\rho\sigma}] = \mathrm{i}(W^{\mu\rho}g^{\nu\sigma} - W^{\mu\sigma}g^{\nu\rho} + W^{\nu\sigma}g^{\mu\rho} - W^{\nu\rho}g^{\mu\sigma}),\\
&[P^\mu, W^{\rho\sigma}] = 0, \qquad [D, W^{\rho\sigma}] = 0, \qquad [M, W^{\rho\sigma}] = 0,\\
&[W^{\mu\nu}, W^{\rho\sigma}] = \mathrm{i}\left(W^{\mu\rho}\left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2}\right) - W^{\mu\sigma}\left(g^{\nu\rho} - \frac{P^\nu P^\rho}{P^2}\right)\right.\\
&\qquad\qquad\left. + W^{\nu\sigma}\left(g^{\mu\rho} - \frac{P^\mu P^\rho}{P^2}\right) - W^{\nu\rho}\left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2}\right)\right),\\
&[R^\mu, W^{\rho\sigma}] = -\mathrm{i}\frac{P^\rho W^{\sigma\mu} - P^\sigma W^{\rho\mu}}{P^2}.
\end{aligned} \tag{2.7}$$

Equations (2.7) describe an *off-mass-shell relativistic particle* and have been found by explicit calculation for several physical systems (Hanson and Regge 1974, Casalbuoni 1976 (see also Almond 1981b, Brink and Schwarz 1981), Mukunda *et al* 1980, Almond 1981a, 1982a). The invariants of the algebra are M, giving the on-mass-shell mass, $\frac{1}{2}W^{\mu\nu}W_{\mu\nu}$, which for $d \leq 4$ gives the spin of the particle (for $d > 4$, there are other spin invariants; e.g. for $d=5$, $\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma\tau}W_{\mu\nu}W_{\rho\sigma}P_\tau/(P^2)^{1/2}$ is also invariant), and sign (P^2) (for sign $(P^2) = +1$, sign (P^0) is also an invariant). The Weyl group generators $M^{\mu\nu}$ and D can be expressed in terms of R^μ, P^μ and $W^{\mu\nu}$ by

$$\begin{aligned}
&M^{\mu\nu} = P^\mu R^\nu - P^\nu R^\mu + W^{\mu\nu},\\
&D = \tfrac{1}{2}[R^\mu, P_\mu]_+.
\end{aligned} \tag{2.8}$$

We also note the transformation properties of the various operators under a unitary parity operator $\mathscr{P}$, and an anti-unitary time-reversal operator $\mathscr{T}$:

$$\begin{aligned}
&\mathscr{P}M^{\mu\nu}\mathscr{P}^{-1} = \eta(\mu)\eta(\nu)M^{\mu\nu}, \qquad &&\mathscr{T}M^{\mu\nu}\mathscr{T}^{-1} = -\eta(\mu)\eta(\nu)M^{\mu\nu},\\
&\mathscr{P}P^\mu\mathscr{P}^{-1} = \eta(\mu)P^\mu, &&\mathscr{T}P^\mu\mathscr{T}^{-1} = \eta(\mu)P^\mu,\\
&\mathscr{P}D\mathscr{P}^{-1} = D, &&\mathscr{T}D\mathscr{T}^{-1} = -D,\\
&\mathscr{P}R^\mu\mathscr{P}^{-1} = \eta(\mu)R^\mu, &&\mathscr{T}R^\mu\mathscr{T}^{-1} = -\eta(\mu)R^\mu,
\end{aligned} \tag{2.9}$$

$$\mathcal{P}W^{\mu\nu}\mathcal{P}^{-1}=\eta(\mu)\eta(\nu)W^{\mu\nu}, \qquad \mathcal{T}W^{\mu\nu}\mathcal{T}^{-1}=-\eta(\mu)\eta(\nu)W^{\mu\nu},$$
$$\mathcal{P}M\mathcal{P}^{-1}=M, \qquad \mathcal{T}M\mathcal{T}^{-1}=M,$$

where $\eta(0)=+1$, $\eta(\mathrm{i})=-1$.

For the case sign $(P^2)=+1$, the spin operator is defined by

$$S^{ij}(P,W)=-L^{-1}(P)^i{}_\rho L^{-1}(P)^j{}_\sigma W^{\rho\sigma}=-W^{ij}+\frac{P^iW^{0j}-P^jW^{0i}}{(P^2)^{1/2}+P^0}, \tag{2.10}$$

where $L^{-1}(P)^\mu{}_\nu$ is the matrix operator

$$L^{-1}(P)^\mu{}_\nu=\begin{pmatrix} \dfrac{P^0}{(P^2)^{1/2}} & \dfrac{P_j}{(P^2)^{1/2}} \\ -\dfrac{P^i}{(P^2)^{1/2}} & \delta^i{}_j-\dfrac{P^iP_j}{(P^2)^{1/2}((P^2)^{1/2}+P^0)} \end{pmatrix} \tag{2.11}$$

which takes P^ν into $((P^2)^{1/2},\mathbf{0})$. The S^{ij} generate the Lie algebra of the $(d-1)$-dimensional rotation group $\mathrm{SO}(d-1)$,

$$[S^{ij},S^{kl}]=\mathrm{i}(S^{ik}\delta^{jl}-S^{il}\delta^{jk}+S^{jl}\delta^{ik}-S^{jk}\delta^{il}), \tag{2.12}$$

and we can invert equation (2.10) to express $W^{\mu\nu}$ in terms of S^{ij}:

$$W^{0i}=P^jS^{ij}/(P^2)^{1/2}, \tag{2.13a}$$

$$W^{ij}=-\left(S^{ij}-\frac{P^iP^lS^{jl}-P^jP^lS^{il}}{(P^2)^{1/2}((P^2)^{1/2}+P^0)}\right). \tag{2.13b}$$

We also note the commutators

$$\begin{aligned} &\left[R^i+\frac{W^{0i}}{(P^2)^{1/2}+P^0},S^{kl}\right]=0, \qquad [R^0,S^{kl}]=0,\\ &\left[R^0,R^i+\frac{W^{0i}}{(P^2)^{1/2}+P^0}\right]=0,\\ &\left[R^i+\frac{W^{0i}}{(P^2)^{1/2}+P^0},R^j+\frac{W^{0j}}{(P^2)^{1/2}+P^0}\right]=0, \end{aligned} \tag{2.14}$$

which, together with $[R^\mu,P^\nu]=-\mathrm{i}g^{\mu\nu}$ and $[S^{ij},P^\mu]=0$, allow us to write the canonical form

$$\begin{aligned} &P^\mu\to p^\mu, \qquad S^{kl}\to S^{kl}(p), \qquad R^0\to-\mathrm{i}\left(\frac{\partial}{\partial p^0}\right)_{p^i},\\ &R^i_{\mathrm{C}}\equiv R^i+\frac{W^{0i}}{(P^2)^{1/2}+P^0}\to\mathrm{i}\left(\frac{\partial}{\partial p^i}\right)_{p^0,p^{(j\neq 1)}}, \end{aligned} \tag{2.15}$$

where $S^{kl}(p)=-L^{-1}(p)^k{}_\rho L^{-1}(p)^l{}_\sigma W^{\rho\sigma}$ acting on a state of momentum p^μ is given by a matrix irreducible representation of the generators of $\mathrm{SO}(d-1)$, and the p-derivatives are taken at constant $S^{kl}(p)$. We shall henceforth refer to R^i_{C} as the 'canonical' position operator.

Now let us write the algebra, equations (2.4), in light-cone coordinates. Since, in this paper, we shall be concerned only with states of a single massless particle for which M has the sole eigenvalue zero, we shall not write this operator explicitly from now on.

We define the light-cone operators by

$$\begin{aligned}
&P^{\pm}=(P^0\pm P^{(d-1)})/\sqrt{2}, \qquad P^i_{\mathrm{T}}=P^i \qquad (i=1\ldots(d-2)),\\
&M^{\pm i}=(M^{0i}\pm M^{(d-1)i})/\sqrt{2} \qquad (i=1\ldots(d-2)),\\
&M^{ij}_{\mathrm{T}}=M^{ij} \qquad (i,j=1\ldots(d-2)),\\
&M^{+-}=-M^{-+}=-M^{0(d-1)},
\end{aligned} \tag{2.16}$$

with similar expressions for the components of R^{μ} and $W^{\mu\nu}$. Henceforth, we shall drop the 'T' subscripts, and latin italic indices will be understood to run from 1 to $(d-2)$. Similarly, boldface type will denote a transverse vector of $(d-2)$ components.

On writing the Lie algebra, equations (2.4), in terms of these operators, we immediately see that the sets $(M^{ij}, -M^{+i}, P^i, P^-, D+M^{0(d-1)}, P^+)$ and $(M^{ij}, -M^{-i}, P^i, P^+, D-M^{0(d-1)}, P^-)$ each form a non-relativistic algebra $(J^{ij}, K^i, P^i, H, D, M)$, of the kind discussed in appendix 1, in $\Delta=(d-2)$ space dimensions, together with the extra commutators

$$\begin{aligned}
&[M^{+i}, M^{-j}]=\mathrm{i}(M^{ij}+M^{0(d-1)}\delta^{ij}),\\
&[D+M^{0(d-1)}, M^{-i}]=-\mathrm{i}M^{-i}\\
&[D-M^{0(d-1)}, M^{+i}]=-\mathrm{i}M^{+i}.
\end{aligned} \tag{2.17}$$

Let us call these two 'non-relativistic' algebras A and B respectively, and calculate their 'spin', 'time' and 'position' operators given by equations (A1.4). On using equations (2.5), (2.6) and (2.8), we find

$$S^{ij}_A=-W^{ij}-(W^{+i}P^j-W^{+j}P^i)/P^+, \tag{2.18a}$$

$$T_A=R^-, \tag{2.18b}$$

$$R^i_A=R^i+W^{+i}/P^+ \tag{2.18c}$$

and

$$S^{ij}_B=-W^{ij}-(W^{-i}P^j-W^{-j}P^i)/P^-, \tag{2.19a}$$

$$T_B=R^+, \tag{2.19b}$$

$$R^i_B=R^i+W^{-i}/P^-. \tag{2.19c}$$

We shall henceforth, for brevity, denote the operators $D\pm M^{0(d-1)}$ by D_A and D_B, respectively. The operators of equations (2.18) and (2.19) transform under parity and time-reversal as

$$\begin{aligned}
&\mathscr{P}S^{ij}_A\mathscr{P}^{-1}=S^{ij}_B, \qquad \mathscr{T}S^{ij}_A\mathscr{T}^{-1}=-S^{ij}_B,\\
&\mathscr{P}T_A\mathscr{P}^{-1}=T_B, \qquad \mathscr{T}T_A\mathscr{T}^{-1}=-T_B,\\
&\mathscr{P}R^i_A\mathscr{P}^{-1}=-R^i_B, \qquad \mathscr{T}R^i_A\mathscr{T}^{-1}=R^i_B,
\end{aligned} \tag{2.20}$$

so that $\mathscr{P}$ and $\mathscr{T}$ each take one light-cone algebra into the other. We also note the transformation properties of the 'dilatation' generators,

$$\mathscr{P}D_A\mathscr{P}^{-1}=D_B, \tag{2.21a}$$

$$\mathscr{T}D_A\mathscr{T}^{-1}=-D_B, \tag{2.21b}$$

which are of interest since these operators generate the light-cone transformations which will be discussed in §§ 3 and 4.

3. Off-mass-shell massless particles (general d)

We now study the infinite transformations generated by D_A and D_B. These transformations are of the same form as the Gartenhaus–Schwartz transformation (Gartenhaus and Schwartz 1957, Osborn 1968, Close and Copley 1970) which is used in constructing both internal operators and an explicitly translation-invariant wavefunction for both non-relativistic (Gartenhaus and Schwartz 1957) and relativistic (Osborn 1968, Close and Copley 1970) many-particle systems. We have, in fact, shown that the Gartenhaus–Schwartz transformation is nothing but an infinite centre-of-mass dilatation (Almond 1973a, §§ II.3 and III.3). Throughout this section, we shall constantly use the equation $P_\mu W^{\mu\nu} = 0$ expressed in component form

$$\begin{aligned} &P^+ W^{-j} + P^- W^{+j} - P^i W^{ij} = 0, \\ &P^i W^{+i} + P^+ W^{-+} = 0, \\ &P^i W^{-i} - P^- W^{-+} = 0. \end{aligned} \tag{3.1}$$

We shall also assume, throughout §§ 3 and 4, that sign $(P^2) = +1$ and sign $(P^0) = +1$, which automatically means that sign $(P^+) = +1$ and sign $(P^-) = +1$.

3.1. *The infinite transformation generated by D_A: effect on operators*

This section is in two parts. In § 3.1.1 we find the transformed momentum, spin and position operators, and the commutation relations which they satisfy, and give a canonical representation of them. In § 3.1.2 we express the Weyl group generators in terms of the transformed operators, and give four different canonical representations of them: (i) off-mass-shell with $(p^+, p^-, \boldsymbol{p})$ as independent variables; (ii) off-mass-shell with $(p^2, p^+, \boldsymbol{p})$ as independent variables; (iii) on-mass-shell, i.e. $p^2 = 0$ with $(p^+, \boldsymbol{p})$ as independent variables; (iv) on-mass-shell, i.e. $p^2 = 0$; with $(p^{(d-1)}, \boldsymbol{p})$ as independent variables.

3.1.1. Momentum, spin, and position operators. First of all, we note the transformation properties of P^μ, R^μ and $W^{\mu\nu}$ under a *finite* 'dilatation' generated by D_A:

$$\begin{aligned} &e^{-i\alpha D_A} P^+ e^{i\alpha D_A} = P^+, \\ &e^{-i\alpha D_A} P^- e^{i\alpha D_A} = e^{-2\alpha} P^-, \\ &e^{-i\alpha D_A} P^i e^{i\alpha D_A} = e^{-\alpha} P^i, \\ &e^{-i\alpha D_A} R^+ e^{i\alpha D_A} = e^{2\alpha} R^+, \\ &e^{-i\alpha D_A} R^- e^{i\alpha D_A} = R^-, \\ &e^{-i\alpha D_A} R^i e^{i\alpha D_A} = e^{\alpha} R^i, \\ &e^{-i\alpha D_A} W^{\pm i} e^{i\alpha D_A} = e^{\pm\alpha} W^{\pm i}, \\ &e^{-i\alpha D_A} W^{ij} e^{i\alpha D_A} = W^{ij}, \\ &e^{-i\alpha D_A} W^{-+} e^{i\alpha D_A} = W^{-+}. \end{aligned} \tag{3.2}$$

On using these equations and the inverses of equations (2.16),

$$
\begin{aligned}
P^0 &= (P^+ + P^-)/\sqrt{2}, & P^{(d-1)} &= (P^+ - P^-)/\sqrt{2},\\
R^0 &= (R^+ + R^-)/\sqrt{2}, & R^{(d-1)} &= (R^+ - R^-)/\sqrt{2},\\
W^{0i} &= (W^{+i} + W^{-i})/\sqrt{2}, & W^{(d-1)i} &= (W^{+i} - W^{-i})/\sqrt{2},
\end{aligned}
\tag{3.3}
$$

we find that, under an infinite D_A transformation, the various components of the 'canonical' position operator transform as

$$
\begin{aligned}
&\lim_{\alpha\to\infty} e^{-2\alpha}\, e^{-i\alpha D_A}(1/\sqrt{2})(R^0 + R_C^{(d-1)})\, e^{i\alpha D_A} = R^+,\\
&\lim_{\alpha\to\infty} e^{-i\alpha D_A}(1/\sqrt{2})(R^0 - R_C^{(d-1)})\, e^{i\alpha D_A} = R^- - W^{-+}/P^+ \equiv R_A^-,\\
&\lim_{\alpha\to\infty} e^{-\alpha}\, e^{-i\alpha D_A} R_C^i\, e^{i\alpha D_A} = R^i + W^{+i}/P^+ = R_A^i,
\end{aligned}
\tag{3.4}
$$

and the various components of the spin operator, given by equations (2.10) and (2.11), transform as

$$
\begin{aligned}
&\lim_{\alpha\to\infty} e^{-i\alpha D_A} S^{ij}\, e^{i\alpha D_A} = -W^{ij} - (W^{-i}P^j - W^{+j}P^i)/P^+ = S_A^{ij},\\
&\lim_{\alpha\to\infty} e^{-i\alpha D_A} S^{(d-1)k}\, e^{i\alpha D_A} = -(P^2)^{1/2} W^{+k}/P^+ \equiv S_A^{(d-1)k},
\end{aligned}
\tag{3.5}
$$

where, it will be recalled, $R^+(=T_A)$, R_A^i and S_A^{ij} are the time, position and spin operators of equations (2.18). To find their commutation relations with one another, we can either apply the infinite D_A transformation to equations (2.12) and (2.14), or evaluate them directly using equations (2.7) and (3.1). We find

$$
\begin{aligned}
&[S_A^{ij}, S_A^{kl}] = i(S_A^{ik}\delta^{jl} - S_A^{il}\delta^{jk} + S_A^{jl}\delta^{ik} - S_A^{jk}\delta^{il}),\\
&[S_A^{(d-1)k}, S_A^{(d-1)l}] = iS_A^{kl},\\
&[S_A^{ij}, S_A^{(d-1)k}] = -i(S_A^{(d-1)i}\delta^{jk} - S_A^{(d-1)j}\delta^{ik}),\\
&[P^\mu, S_A^{ij}] = [P^\mu, S_A^{(d-1)k}] = 0,\\
&[R^+, S_A^{jk}] = [R_A^-, S_A^{jk}] = [R_A^i, S_A^{jk}] = 0,\\
&[R^+, S_A^{(d-1)k}] = [R_A^-, S_A^{(d-1)k}] = [R_A^i, S_A^{(d-1)k}] = 0,\\
&[R^+, R_A^-] = [R^+, R_A^i] = [R_A^-, R_A^i] = [R_A^i, R_A^j] = 0,\\
&[R^+, P^-] = -i, \qquad [R_A^-, P^+] = -i, \qquad [R_A^i, P^j] = i\delta^{ij},\\
&[R^+, P^i] = [R_A^-, P^i] = [R^+, P^+] = 0,\\
&[R_A^i, P^+] = [R_A^-, P^-] = [R_A^i, P^-] = 0,
\end{aligned}
\tag{3.6}
$$

which allow us to write the canonical representation

$$
\begin{aligned}
&P^\mu \to p^\mu, \qquad S_A^{ij} \to S_A^{ij}(p), \qquad S_A^{(d-1)k} \to S_A^{(d-1)k}(p),\\
&R^+ \to -i\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^i}, \qquad R_A^- \to -i\left(\frac{\partial}{\partial p^+}\right)_{p^-,p^i}, \qquad R_A^i \to i\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{j\neq i}}
\end{aligned}
\tag{3.7}
$$

where $(S_A^{ij}(p), S_A^{(d-1)k}(p))$ are the matrix irreducible representation of the generators of SO$(d-1)$ when acting on a state of momentum p^μ, and where the derivatives are taken at constant $S_A^{ij}(p)$ and $S_A^{(d-1)k}(p)$.

3.1.2. Weyl group generators in terms of the transformed operators. Let us now express the Weyl group generators $M^{\mu\nu}$ and D in terms of the transformed operators by equations (2.8):

$$\begin{aligned} &M^{+i} = P^+R_A^i - P^iR^+, \qquad M^{ij} = P^iR_A^j - P^jR_A^i - S_A^{ij}, \\ &M^{-i} = P^-R_A^i - P^iR_A^- + P^jS_A^{ij}/P^+ + (P^2)^{1/2}S_A^{(d-1)i}/P^+, \\ &D_A = [P^-, R^+]_+ - \tfrac{1}{2}[P^i, R_A^i]_+, \qquad D_B = [P^+, R_A^-]_+ - \tfrac{1}{2}[P^i, R_A^i]_+, \end{aligned} \tag{3.8}$$

where, in evaluating the expressions for M^{-i}, D_A and D_B, we have used equations (3.1). Using equations (3.7), we can write a canonical representation for these operators:

$$\begin{aligned} &M^{+i} \to \mathrm{i}p^+\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} + \mathrm{i}p^i\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^k}, \\ &M^{ij} \to \mathrm{i}p^i\left(\frac{\partial}{\partial p^j}\right)_{p^+,p^-,p^{k(\neq j)}} - \mathrm{i}p^j\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} - S_A^{ij}(p), \\ &M^{-i} \to \mathrm{i}p^-\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} + \mathrm{i}p^i\left(\frac{\partial}{\partial p^+}\right)_{p^-,p^k} + \frac{p^jS_A^{ij}(p)}{p^+} + \frac{(p^2)^{1/2}S_A^{(d-1)i}(p)}{p^+}, \\ &D_A \to -\mathrm{i}\left(2p^-\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^k} + p^i\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} + \frac{d}{2}\right) \\ &D_B \to -\mathrm{i}\left(2p^+\left(\frac{\partial}{\partial p^+}\right)_{p^-,p^k} + p^i\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} + \frac{d}{2}\right). \end{aligned} \tag{3.9}$$

We immediately note that M^{-i} automatically gives zero when acting on a state with momentum $(p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2})$, in contrast to the Poincaré group (Wigner 1939, Weinberg 1964a) where this condition has to be put in by hand. Similarly M^{+i} automatically gives zero when acting on a state with momentum $(p^-/\sqrt{2}, \mathbf{0}, -p^-/\sqrt{2})$. We also note that, for *any* state with light-like momentum, the $S_A^{(d-1)i}$ drop out of the expression for M^{-i} in equations (3.8) (similarly for $S_A^{(d-1)i}(p)$ in the canonical representation for M^{-i} in equations (3.9)). This ties in with the fact that the spin 'little group' of the massless irreducible representations of the d-dimensional Poincaré group is SO$(d-2)$ rather than SO$(d-1)$.

We can, in fact, construct a canonical form for the generators which is explicitly valid for $p^2=0$ by changing variables from $(p^+, p^-, \boldsymbol{p})$ to $(p^2, p^+, \boldsymbol{p})$ and then putting $p^2=0$. The expressions for the partial derivatives are

$$\begin{aligned} &\left(\frac{\partial}{\partial p^+}\right)_{p^2,p^k} = \left(\frac{\partial}{\partial p^+}\right)_{p^-,p^k} - \frac{p^-}{p^+}\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^k}, \qquad \left(\frac{\partial}{\partial p^2}\right)_{p^+,p^k} = \frac{1}{2p^+}\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^k} \\ &\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^+,p^{k(\neq i)}} = \left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{k(\neq i)}} + \frac{p^i}{p^+}\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^k}, \end{aligned} \tag{3.10}$$

and, on substituting equations (3.10) into equations (3.9), we find the $(p^2, p^+, \boldsymbol{p})$

representation

$$M^{+i} \to \mathrm{i}p^{+}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}},$$

$$M^{ij} \to \mathrm{i}p^{i}\left(\frac{\partial}{\partial p^{j}}\right)_{p^2,p^+,p^{k(\neq j)}} - \mathrm{i}p^{j}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} - S_A^{ij}(p),$$

$$M^{-i} \to \frac{\mathrm{i}(\boldsymbol{p}^2+p^2)}{2p^+}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \mathrm{i}p^{i}\left(\frac{\partial}{\partial p^{+}}\right)_{p^2,p^k} + \frac{p^j S_A^{ij}(p)}{p^+} + \frac{(p^2)^{1/2} S_A^{(d-1)i}(p)}{p^+}, \tag{3.11}$$

$$D_A \to -\mathrm{i}\left(2p^2\left(\frac{\partial}{\partial p^2}\right)_{p^+,p^k} + p^{i}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \frac{d}{2}\right),$$

$$D_B \to -\mathrm{i}\left(2p^2\left(\frac{\partial}{\partial p^2}\right)_{p^+,p^k} + 2p^{+}\left(\frac{\partial}{\partial p^{+}}\right)_{p^2,p^k} + p^{i}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \frac{d}{2}\right).$$

The representation for $p^2 = 0$ is found by noting that

$$\delta(p^2)2p^2(\partial/\partial p^2)\delta(p^2) = \delta(p^2)(p^2(\vec{\partial}/\partial p^2) - (\overleftarrow{\partial}/\partial p^2)p^2 - 1)\delta(p^2) = (-1)\delta(p^2)\delta(p^2)$$

and substituting into equations (3.11):

$$M^{+i} \to \mathrm{i}p^{+}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}},$$

$$M^{ij} \to \mathrm{i}p^{i}\left(\frac{\partial}{\partial p^{j}}\right)_{p^2,p^+,p^{k(\neq j)}} - \mathrm{i}p^{j}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} - S_A^{ij}(p),$$

$$M^{-i} \to \mathrm{i}\,\frac{\boldsymbol{p}^2}{2p^+}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \mathrm{i}p^{i}\left(\frac{\partial}{\partial p^{+}}\right)_{p^2,p^k} + \frac{p^j S_A^{ij}(p)}{p^+}, \tag{3.12}$$

$$D_A \to -\mathrm{i}\left(p^{i}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \frac{d-2}{2}\right)$$

$$D_B \to -\mathrm{i}\left(2p^{+}\left(\frac{\partial}{\partial p^{+}}\right)_{p^2,p^k} + p^{i}\left(\frac{\partial}{\partial p^{i}}\right)_{p^2,p^+,p^{k(\neq i)}} + \frac{d-2}{2}\right).$$

We can find another canonical form valid for $p^2 = 0$ by eliminating p^0 in favour of p^i and $p^{(d-1)}$ First we change variables from $(p^+, p^-, \boldsymbol{p})$ to $(p^0, \boldsymbol{p}, p^{(d-1)})$,

$$\left(\frac{\partial}{\partial p^{0}}\right)_{p^{(d-1)},p^k} = \frac{1}{\sqrt{2}}\left(\left(\frac{\partial}{\partial p^{+}}\right)_{p^-,p^k} + \left(\frac{\partial}{\partial p^{-}}\right)_{p^+,p^k}\right)$$

$$\left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^0,p^k} = \frac{1}{\sqrt{2}}\left(\left(\frac{\partial}{\partial p^{+}}\right)_{p^-,p^k} - \left(\frac{\partial}{\partial p^{-}}\right)_{p^+,p^k}\right) \tag{3.13a}$$

$$\left(\frac{\partial}{\partial p^{i}}\right)_{p^0,p^{(d-1)},p^{k(\neq i)}} = \left(\frac{\partial}{\partial p^{i}}\right)_{p^+,p^-,p^{k(\neq i)}},$$

and then change variables again from $(p^0, \boldsymbol{p}, p^{(d-1)})$ to $(p^2, \boldsymbol{p}, p^{(d-1)})$:

$$\begin{aligned}
\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^{(d-1)},p^{k(\neq i)}} &= \left(\frac{\partial}{\partial p^i}\right)_{p^0,p^{(d-1)},p^{k(\neq i)}} + \frac{p^i}{p^0}\left(\frac{\partial}{\partial p^0}\right)_{p^{(d-1)},p^k},\\
\left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^2,p^k} &= \left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^0,p^k} + \frac{p^{(d-1)}}{p^0}\left(\frac{\partial}{\partial p^0}\right)_{p^{(d-1)},p^k},\\
\left(\frac{\partial}{\partial p^2}\right)_{p^{(d-1)},p^k} &= \frac{1}{2p^0}\left(\frac{\partial}{\partial p^0}\right)_{p^{(d-1)},p^k}.
\end{aligned} \tag{3.13b}$$

We then find, on applying equations (3.13), together with $p^2=0$, to equations (3.9),

$$\begin{aligned}
M^{ij} &\to \mathrm{i}p^i\left(\frac{\partial}{\partial p^j}\right)_{p^2,p^{(d-1)},p^{k(\neq j)}} - \mathrm{i}p^j\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^{(d-1)},p^{k(\neq i)}} - S_{\mathrm{A}}^{ij}(p),\\
M^{0i} &\to \mathrm{i}p^0\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^{(d-1)},p^{k(\neq i)}} + \frac{p^j S_{\mathrm{A}}^{ij}(p)}{p^0+p^{(d-1)}},\\
M^{(d-1)i} &\to \mathrm{i}p^{(d-1)}\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^{(d-1)},p^{k(\neq i)}} - \mathrm{i}p^i\left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^2,p^k} - \frac{p^j S_{\mathrm{A}}^{ij}(p)}{p^0+p^{(d-1)}},\\
M^{0(d-1)} &\to \mathrm{i}p^0\left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^2,p^k},\\
D &\to -\mathrm{i}\left(p^i\left(\frac{\partial}{\partial p^i}\right)_{p^2,p^{(d-1)},p^{k(\neq i)}} + p^{(d-1)}\left(\frac{\partial}{\partial p^{(d-1)}}\right)_{p^2,p^k} + \frac{d-2}{2}\right)
\end{aligned} \tag{3.14}$$

where p^0 is shorthand for $(p^2+(p^{(d-1)})^2)^{1/2}$. More will be said about this canonical representation for the special case $d=4$ which will be discussed in § 4.1, where we shall show that this form of the Lorentz transformation generators $M^{\mu\nu}$ is just that given for massless irreducible representations of the Poincaré group by Lomont and Moses (1962) and Chakrabarti (1966).

3.2. The infinite transformation generated by D_{A}: effect on states

The *off-mass-shell states* of a massless particle transform as a unitary irreducible representation of the Weyl group. The irreducible representations are labelled by the mass $m=0$ and the invariants of the $(d-1)$-dimensional rotation group SO$(d-1)$ which we shall call s_i ($i=1\ldots L$, where L, the rank of SO$(d-1)$, is given by $L=\frac{1}{2}d-1$ for d even and by $L=\frac{1}{2}(d-1)$ for d odd). States within an irreducible representation are labelled by L mutually commuting spin components. However, this is not the whole story. For $d\geqslant 6$, there are $\frac{1}{2}[\frac{1}{2}(d-1)(d-2)-3L]$ extra labelling operators (analogous to isospin in the case of SU(3))†. Futhermore, since we shall be going onto the mass-shell, where only the spin components S_{A}^{ij} are physically significant (see discussion following equations (3.9)), it would clearly be unwise to use the $S^{(d-1)k}$ in labelling the states. We shall therefore label the states within an irreducible representation by the eigenvalues of P^μ, and the mutually commuting S^{ij} which we shall call p^μ and σ_l, respectively, by the spin invariants in the $(d-2)$-dimensional transverse space which we shall call t_j, and by the extra labelling operators in the $(d-2)$-dimensional transverse space (for $d\geqslant 7$) which we shall call u_k. Both j and l run from 1 to K, where K, the rank of SO$(d-2)$,

† A Lie group with a parameters and of rank b has $\frac{1}{2}(a-3b)$ extra labelling operators.

is equal to L for even d, and equal to $L-1$ for odd d. The index k runs from 1 to $\frac{1}{2}[\frac{1}{2}(d-2)(d-3)-3K]$. The total number of operators labelling states within an irreducible representation is $K+K+\frac{1}{2}[\frac{1}{2}(d-2)(d-3)-3K]$ which is equal to $L+\frac{1}{2}[\frac{1}{2}(d-1)(d-2)-3L]$, the total number of 'conventional' labelling operators.

An off-mass-shell state of a massless particle is therefore written $|o, s; p, t, u, \sigma\rangle$ (where s denotes $(s_1, s_2, \ldots, s_L)$; similarly for t, u, σ), and states within an irreducible representation are normalised to

$$\langle o, s; p', t', u', \sigma'|o, s; p, t, u, \sigma\rangle = \delta_{t't}\delta_{u'u}\delta_{\sigma'\sigma}\delta^d(p'-p), \tag{3.15}$$

where $\delta_{t't} = \delta_{t'_1t_1} \ldots \delta_{t'_Kt_K}$; similarly for $\delta_{u'u}$, $\delta_{\sigma'\sigma}$. The effect on a single-particle state of a unitary operator of the Weyl group

$$U(\lambda, L, a) = e^{ia\cdot P}\, e^{i\frac{1}{2}\alpha_{\mu\nu}M^{\mu\nu}}\, e^{i(\ln\lambda)D} \tag{3.16}$$

is given by (cf Almond 1973a, equation (III.64))

$$U(\lambda, L, a)|o, s; p, t, u, \sigma\rangle$$
$$= \lambda^{-d/2}\, e^{ip'\cdot a}\sum_{t'}\sum_{u'}\sum_{\sigma'}|o, s; p', t', u', \sigma'\rangle D^s_{t'u'\sigma'tu\sigma}(L^{-1}(p')LL(p)), \tag{3.17}$$

where $p' = \lambda^{-1}Lp$, $L^{-1}(p)$ is defined in equation (2.11), and $D^s_{t'u'\sigma'tu\sigma}(R)$ is the unitary irreducible representation of SO$(d-1)$.

We are now going to construct *on-mass-shell states* by considering the effect, on an off-mass-shell particle state, of the operator λ^{iD_A} with $\lambda\to\infty$. Since equation (3.17) is clearly very complicated for general d, we shall restrict ourselves to states of an off-mass-shell particle moving in the $(d-1)$-direction, i.e. $|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle$. For this case, $L^{-1}(p')LL(p)$ is just the unit matrix, and $D^s_{t'u'\sigma'tu\sigma}(L^{-1}(p')LL(p))$ is just equal to $\delta_{t't}\delta_{u'u}\delta_{\sigma'\sigma}$, so the *finite* transformation just reads

$$\lambda^{iD_A}|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle$$
$$= \lambda^{-d/2}|o, s; ((p^+ + \lambda^{-2}p^-)/\sqrt{2}, \mathbf{0}, (p^+ - \lambda^{-2}p^-)/\sqrt{2}), t, u, \sigma\rangle. \tag{3.18}$$

We now define the on-mass-shell state by

$$|o, s, t; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), u, \sigma\}$$
$$= (1/\sqrt{\delta(0)})\lim_{\lambda\to\infty}\lambda^{d/2}\lambda^{iD_A}|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle, \tag{3.19}$$

where we have used the notation $|\}$ rather than $|\rangle$ to denote that the on-mass-shell states are in a different Hilbert space to the off-mass-shell states since the infinite transformation is non-unitary (it has no inverse)†. The t_j now label the irreducible representations rather than the states within an irreducible representation. The $1/\sqrt{\delta(0)}$ in equation (3.19) is familiar from the Gartenhaus–Schwartz transformation (Miglietta 1970, Palumbo 1971, Ernst *et al* 1973, Malecki and Picchi 1975) and occurs because, in going onto the mass-shell, we are losing the degree of freedom p^-. In fact the states of

† Note the interesting possibility that there exist in the $p^2=0$ Hilbert space, particles that are *not* the $p^2\to 0$ limit of particles in the $p^2>0$, $p^0>0$ Hilbert space. We call such particles 'conons' as distinct from the 'luxons' considered here. See also Almond (1982a,b).

equation (3.19) are normalised according to

$$\{o, s, t; (p'^{+}/\sqrt{2}, \mathbf{0}, p'^{+}/\sqrt{2}), u', \sigma'|o, s, t; (p^{+}/\sqrt{2}, \mathbf{0}, p^{+}/\sqrt{2}), u, \sigma\}$$
$$= \delta_{u'u}\delta_{\sigma'\sigma}2p^{+}\delta(p'^{+}-p^{+})\delta^{d-2}(\mathbf{0})$$
$$= \delta_{u'u}\delta_{\sigma'\sigma}2p^{0}\delta(p'^{(d-1)}-p^{(d-1)})\delta^{d-2}(\mathbf{0}). \tag{3.20}$$

To show more clearly the physical meaning of the eigenvalues s_i, t_j, u_k and σ_l of the spin invariants and spin components in $|o, s, t; (p^{+}/\sqrt{2}, \mathbf{0}, p^{+}/\sqrt{2}), u, \sigma\}$, we express S_A^{ij} and $S_A^{(d-1)k}$ (via their definitions, equations (3.5)) in terms of S^{ij} and $S^{(d-1)k}$ using equations (2.13) and find

$$S_A^{ij} = S^{ij} + \frac{P^iP^kS^{jk} - P^jP^kS^{ik}}{\sqrt{2}P^{+}((P^2)^{1/2}+P^0)} - \frac{(P^iS^{(d-1)j} - P^jS^{(d-1)i})((P^2)^{1/2}+\sqrt{2}P^{+})}{\sqrt{2}P^{+}((P^2)^{1/2}+P^0)} \tag{3.21a}$$

$$S_A^{(d-1)k} = S^{(d-1)k} - \frac{(P^2\delta^{ik} - P^iP^k)S^{(d-1)i}}{\sqrt{2}P^{+}((P^2)^{1/2}+P^0)} - \frac{((P^2)^{1/2}+\sqrt{2}P^{+})P^iS^{ki}}{\sqrt{2}P^{+}((P^2)^{1/2}+P^0)}. \tag{3.21b}$$

Since S_A^{ij} and $S_A^{(d-1)k}$ commute with D_A, we see that equations (3.18) and (3.20) give

$$S_A^{ij}|o, s, t; (p^{+}/\sqrt{2}, \mathbf{0}, p^{+}/\sqrt{2}), u, \sigma\}$$
$$= (1/\sqrt{\delta(0)}) \lim_{\lambda\to\infty} \lambda^{d/2}\lambda^{\mathrm{i}D_A}S^{ij}|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle, \tag{3.22a}$$

$$S_A^{(d-1)k}|o, s, t; (p^{+}/\sqrt{2}, \mathbf{0}, p^{+}/\sqrt{2}), u, \sigma\}$$
$$= (1/\sqrt{\delta(0)}) \lim_{\lambda\to\infty} \lambda^{d/2}\lambda^{\mathrm{i}D_A}S^{(d-1)k}|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle, \tag{3.22b}$$

so that the effect of S_A^{ij} and $S_A^{(d-1)k}$ on the on-mass-shell states $|o, s, t; (p^{+}/\sqrt{2}, \mathbf{0}, p^{+}/\sqrt{2}), u, \sigma\}$ is exactly the same as the effect of S^{ij} and $S^{(d-1)k}$ on the off-mass-shell states $|o, s; (p^0, \mathbf{0}, p^{(d-1)}), t, u, \sigma\rangle$.

An on-mass-shell (i.e. $p^2 = 0$) state of a massless particle of off-mass-shell spin s_i and on-mass-shell spin t_j, with general momentum $p^{\mu} = (p^0, \boldsymbol{p}, p^{(d-1)})$ and given eigenvalues u_k, σ_l is generated from a *standard state* $|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}$ (with $\kappa > 0$) by

$$|o, s, t; p, u, \sigma\} = U(1, \mathscr{L}(p), 0)|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}, \tag{3.23}$$

where $\mathscr{L}(p)$ is a product of a boost in the $(d-1)$-direction and a rotation which takes the $(d-1)$-axis into the $(\boldsymbol{p}, p^{(d-1)})$-direction,

$$U(1, \mathscr{L}(p), 0) = \exp(\mathrm{i}\alpha\hat{p}^iM^{(d-1)i})(p^0/\kappa)^{\mathrm{i}M^{0(d-1)}}, \tag{3.24}$$

with $\tan\alpha = |\boldsymbol{p}|/p^{(d-1)} (0 \leq \alpha \leq \pi)$ and $\hat{\boldsymbol{p}}$ a unit vector in the $\boldsymbol{p}$-direction (in other words, we have written $\alpha_{(d-1)i} = \alpha^{(d-1)i} = \alpha\hat{p}^i$). We could instead have used the operator

$$U(\mathscr{W}(p), 0) = (\sqrt{2}\kappa/p^{+})^{(d-2)/4}\exp(\mathrm{i}p^iM^{+i}/p^{+})(\sqrt{2}\kappa/p^{+})^{\mathrm{i}\frac{1}{2}D_B},$$

but the quoted form is the conventional one (Wigner 1939, Weinberg 1964a). We rewrite the first exponential in equation (3.24) using equation (A3.8) and find for equation (3.23)

$$|o, s, t; p, u, \sigma\} = \mathrm{e}^{\mathrm{i}p^iM^{+i}/p^{+}}(p^{+}/\sqrt{2}\kappa)^{\mathrm{i}M^{0(d-1)}}|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}, \tag{3.25}$$

where we have used the fact that M^{-i} gives zero acting on $|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}$, together with the equations $\sqrt{2}\tan(\frac{1}{2}\alpha) = |\boldsymbol{p}|/p^{+}$ and $p^0\cos^2(\frac{1}{2}\alpha) = p^{+}/\sqrt{2}$. (Equation

(3.25) looks as if it might not be well-defined for $p^+=0$ (i.e. $\alpha=\pi$), but in appendix 4 we shall show that the singularity in the first exponential is cancelled by the zero in the second exponential for the states which transform as a suitable finite-dimensional irreducible representation of the homogeneous Lorentz group SO$(d-1, 1)$ with maximum eigenvalue $\mathrm{i}M^{0(d-1)}$.) Since S_A^{ij} and $S_A^{(d-1)k}$ both commute with M^{+i} and $M^{0(d-1)}$ (see equations (3.6) and (3.8)), their effect on the state $|o, s, t; p, u, \sigma\}$ is the same as on the standard state $|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}$. The raising and lowering operators constructed from the S_A^{ij} change the values of u_k and σ_l, while the raising and lowering operators constructed from the $S_A^{(d-1)k}$ change the values of t_j as well. The general on-mass-shell states are normalised to

$$\begin{aligned}\{o, s, t; p', u', \sigma'|o, s, t; p, u, \sigma\}&\\ &=\delta_{u'u}\delta_{\sigma'\sigma}2p^+\delta(p'^+-p^+)\delta^{d-2}(\boldsymbol{p}'-\boldsymbol{p})\\ &=\delta_{u'u}\delta_{\sigma'\sigma}2p^0\delta(p'^{(d-1)}-p^{(d-1)})\delta^{d-2}(\boldsymbol{p}'-\boldsymbol{p}),\end{aligned}\tag{3.26}$$

where the latter equality follows from $p^+=((\boldsymbol{p}^2+(p^{(d-1)})^2)^{1/2}+p^{(d-1)})/\sqrt{2}$.

Let us now work out the effect of a unitary operator of the Weyl group $U(\lambda, L, a)$ on the state $|o, s, t; p, u, \sigma\}$. (We cannot, of course, use equation (3.17) since that is valid only for the off-mass-shell Hilbert space.) The unitary irreducible representations of the $d=4$ Weyl group were first studied by Ottoson (1967), and the subject has been reviewed by Görnitz (1975). On using equation (3.23) and the group property

$$U(\lambda', L', a')U(\lambda, L, a)=U(\lambda'\lambda, L'L, \lambda'L'a+a'),\tag{3.27}$$

we find

$$\begin{aligned}U(\lambda, L, a)|o, s, t; p, u, \sigma\}&\\ &=U(1, \mathscr{L}(\lambda^{-1}Lp), 0)U(\lambda, \mathscr{L}^{-1}(\lambda^{-1}Lp)L\mathscr{L}(p), \mathscr{L}^{-1}(\lambda^{-1}Lp)a)\\ &\quad\times|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}.\end{aligned}\tag{3.28}$$

But $(\lambda, \mathscr{L}^{-1}(\lambda^{-1}Lp)L\mathscr{L}(p), 0)$ leaves $(\kappa, \mathbf{0}, \kappa)$ unchanged, hence $U(\lambda, \mathscr{L}^{-1}(\lambda^{-1}Lp)L\mathscr{L}(p), 0)$ is a unitary operator of the 'little group' whose generators are D_A, M^{-k} and M^{ij} (i.e. the little group is isomorphic to the group of dilatations, displacements and rotations on a $(d-2)$-dimensional Euclidean space):

$$\begin{aligned}&[D_A, P^\mu]|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}=0,\\ &[M^{-k}, P^\mu]|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}=0,\\ &[M^{ij}, P^\mu]|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}=0.\end{aligned}\tag{3.29}$$

In fact, using equations (3.16) and (3.24), we find

$$U(\lambda, \mathscr{L}^{-1}(\lambda^{-1}Lp)L\mathscr{L}(p), 0)=\lambda^{\mathrm{i}D_A}U(1, \mathscr{L}^{-1}(Lp)L\mathscr{L}(p), 0).\tag{3.30}$$

The unitary irreducible representations of the little group with the 'translation' generators represented by zero are given by those of SO$(d-2)$ and are also labelled by the eigenvalue of D_A:

$$M^{-k}|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\} = 0, \tag{3.31a}$$

$$M^{ij}|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\} = -S_A^{ij}|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\}, \tag{3.31b}$$

$$D_A|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\} = (\mathrm{i}\tfrac{1}{2}(d-2)+\rho)|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\}, \tag{3.31c}$$

where ρ is a real number and where $\mathrm{i}\frac{1}{2}(d-2)$ preserves the normalisation equation (3.26). The states $|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}$, which we have been using up to now, all have the same value of ρ which we can take to be zero. We can write

$$U(1, \mathscr{L}^{-1}(Lp)L\mathscr{L}(p), 0) = \exp(\mathrm{i}(\alpha_{-k}M^{-k} + \tfrac{1}{2}\alpha_{ij}M^{ij})), \tag{3.32}$$

where α_{-k} and α_{ij} are functions of $\mathscr{L}^{-1}(Lp)L\mathscr{L}(p)$. On expanding the exponential in equation (3.32), using $[M^{ii}, M^{-k}] = \mathrm{i}(M^{-i}\delta^{jk} - M^{-j}\delta^{ik})$, and equation (3.31*a*), we find

$$U(1, \mathscr{L}^{-1}(Lp)L\mathscr{L}(p), 0)|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\} = \mathrm{e}^{\mathrm{i}\frac{1}{2}\alpha_{ij}M^{ij}}|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\}, \tag{3.33}$$

and, on writing the unitary irreducible representation of SO$(d-2)$ as

$$d^t_{u'\sigma'u\sigma}(R(\mathscr{L}^{-1}(Lp)L\mathscr{L}(p))) = \{t; u', \sigma'|\mathrm{e}^{-\mathrm{i}\frac{1}{2}\alpha_{ij}S_A^{ij}}|t; u, \sigma\}, \tag{3.34}$$

we find, using equations (3.30), (3.31*c*) and (3.34) in equation (3.28), with $|o, s, t; p, u, \sigma\}$ replaced by $|o, s, t, \rho; p, u, \sigma\} = U(1, \mathscr{L}(p), 0)|o, s, t, \rho; (\kappa, \mathbf{0}, \kappa), u, \sigma\}$, that

$$\begin{aligned} U(\lambda, L, a)&|o, s, t, \rho; p, u, \sigma\} \\ &= \lambda^{-\frac{1}{2}(d-2)+\mathrm{i}\rho}\, \mathrm{e}^{\mathrm{i}a\cdot p'} \sum_{u'}\sum_{\sigma'} |o, s, t, \rho; p', u', \sigma'\} \\ &\quad \times d^t_{u'\sigma'u\sigma}(R(\mathscr{L}^{-1}(Lp)L\mathscr{L}(p))), \end{aligned} \tag{3.35}$$

where $p' = \lambda^{-1}Lp$. We shall show elsewhere (Almond 1982b) that ρ is nothing but the evolution parameter on the light-cone (cf Almond 1981a, equation (2.7)). Henceforth, we shall, without loss of generality, put it equal to zero, and use the corresponding states $|o, s, t; p, u, \sigma\}$.

As an example of these formulae, consider $d = 5$. The off-mass-shell states are $|o, s_1, s_2; p, t_1, \sigma_1\rangle$, where the invariants of SO(4) (which has rank $L = 2$) are given by

$$\tfrac{1}{4}S^{ij}S^{ij} + \tfrac{1}{2}S^{4i}S^{4i} = s_1(s_1+1) + s_2(s_2+1),$$

$$\tfrac{1}{2}\varepsilon^{ijk}S^{ij}S^{4k} = s_1(s_1+1) - s_2(s_2+1),$$

and $t_1(t_1+1)$ and σ_1 are the eigenvalues of $\frac{1}{2}S^{ij}S^{ij}$ and S^{12} respectively (which we use instead of S^{12} and S^{43}). On going onto the mass-shell, the spin is described by SO(3) (which has rank $K = 1$) with invariant $\frac{1}{2}S_A^{ij}S_A^{ij} = t_1(t_1+1)$ and with states $|o, s_1, s_2, t_1; p, \sigma_1\}$ labelled by the eigenvalue σ_1 of S_A^{12}.

3.3. *The infinite transformation generated by D_B*

In this section, we shall be very brief and just note the essential differences from the previous two sections.

The transformation properties of P^μ, R^μ and $W^{\mu\nu}$ under a finite 'dilatation' generated by D_B are

$$\begin{aligned}
&e^{-i\alpha D_B}P^+\,e^{i\alpha D_B}=e^{-2\alpha}P^+,\\
&e^{-i\alpha D_B}P^-\,e^{i\alpha D_B}=P^-,\\
&e^{-i\alpha D_B}P^i\,e^{i\alpha D_B}=e^{-\alpha}P^i,\\
&e^{-i\alpha D_B}R^+\,e^{i\alpha D_B}=R^+,\\
&e^{-i\alpha D_B}R^-\,e^{i\alpha D_B}=e^{2\alpha}R^-\\
&e^{-i\alpha D_B}R^i\,e^{i\alpha D_B}=e^{\alpha}R^i,\\
&e^{-i\alpha D_B}W^{\pm i}\,e^{i\alpha D_B}=e^{\mp\alpha}W^{\pm i},\\
&e^{-i\alpha D_B}W^{ij}\,e^{i\alpha D_B}=W^{ij},\\
&e^{-i\alpha D_B}W^{-+}\,e^{i\alpha D_B}=W^{-+}.
\end{aligned}\tag{3.36}$$

Under an infinite transformation, the components of the 'canonical' position operator transform as

$$\begin{aligned}
&\lim_{\alpha\to\infty} e^{-i\alpha D_B}(1/\sqrt{2})(R^0+R_C^{(d-1)})\,e^{i\alpha D_B}=R^++W^{-+}/P^-\equiv R_B^+,\\
&\lim_{\alpha\to\infty} e^{-2\alpha}\,e^{-i\alpha D_B}(1/\sqrt{2})(R^0-R_C^{(d-1)})\,e^{i\alpha D_B}=R^-,\\
&\lim_{\alpha\to\infty} e^{-\alpha}\,e^{-i\alpha D_B}R_C^i\,e^{i\alpha D_B}=R^i+W^{-i}/P^-=R_B^i
\end{aligned}\tag{3.37}$$

and the various components of the spin operator transform as

$$\begin{aligned}
&\lim_{\alpha\to\infty} e^{-i\alpha D_B}S^{ij}\,e^{i\alpha D_B}=-W^{ij}-(W^{-i}P^j-W^{-j}P^i)/P^-=S_B^{ij},\\
&\lim_{\alpha\to\infty} e^{-i\alpha D_B}S^{(d-1)k}\,e^{i\alpha D_B}=(P^2)^{1/2}W^{-k}/P^-\equiv S_B^{(d-1)k},
\end{aligned}\tag{3.38}$$

where $R^-(=T_B)$, R_B^i and S_B^{ij} are the time, position and spin operators of equations (2.19). The different components of the transformed canonical position operator commute with each other and with the transformed spin operators. The spin operators generate the commutator algebra of SO$(d-1)$. We can therefore write the canonical representation

$$\begin{aligned}
&P^\mu\to p^\mu, &&S_B^{ij}\to S_B^{ij}(p), &&S_B^{(d-1)k}\to S_B^{(d-1)k}(p),\\
&R_B^+\to -i\left(\frac{\partial}{\partial p^-}\right)_{p^+,p^i}, &&R^-\to -i\left(\frac{\partial}{\partial p^+}\right)_{p^-,p^i}, &&R_B^i\to i\left(\frac{\partial}{\partial p^i}\right)_{p^+,p^-,p^{j(\neq i)}}
\end{aligned}\tag{3.39}$$

with the p-derivatives taken at constant $S_B^{ij}(p)$ and $S_B^{(d-1)k}(p)$. We can write the generators $M^{\mu\nu}$ and D in terms of the transformed operators†:

† If we were to consider all of Minkowski p-space, instead of just the forward light-cone as in this paper, then the operators of equations (3.8) are valid everywhere except on the plane $p^+=0$ (because of M^{-i}), whilst the operators of equations (3.40) are valid everywhere except on the plane $p^-=0$ (because of M^{+i}).

$$M^{+i}=P^+R^i_B-P^iR^+_B+\frac{P^jS^{ij}_B}{P^-}-\frac{(P^2)^{1/2}S^{(d-1)i}_B}{P^-},$$

$$M^{ij}=P^iR^j_B-P^jR^i_B-S^{ij}_B,\qquad M^{-i}=P^-R^i_B-P^iR^-,\tag{3.40}$$

$$D_A=[P^-,R^+_B]_+-\tfrac{1}{2}[P^i,R^i_B]_+,\qquad D_B=[P^+,R^-]_+-\tfrac{1}{2}[P^i,R^i_B]_+.$$

We define the on-mass-shell state

$$|o,s,t;(p^-/\sqrt{2},\mathbf{0},-p^-/\sqrt{2}),u,\sigma]$$
$$=(1/\sqrt{\delta(0)})\lim_{\lambda\to\infty}\lambda^{d/2}\lambda^{iD_B}|o,s;(p^0,\mathbf{0},p^{(d-1)}),t,u,\sigma\rangle\tag{3.41}$$

and they are normalised to

$$[o,s,t;(p'^-/\sqrt{2},\mathbf{0},-p'^-/\sqrt{2}),u',\sigma'|o,s,t;(p^-/\sqrt{2},\mathbf{0},-p^-/\sqrt{2}),u,\sigma]$$
$$=\delta_{u'u}\delta_{\sigma'\sigma}2p-\delta(p'^--p^-)\delta^{d-2}(\mathbf{0})$$
$$=\delta_{u'u}\delta_{\sigma'\sigma}2p^0\delta(p'^{(d-1)}-p^{(d-1)})\delta^{d-2}(\mathbf{0}).\tag{3.42}$$

Expressing S^{ij}_B and $S^{(d-1)k}_B$ of equations (3.38) in terms of S^{ij} and $S^{(d-1)k}$ using equations (2.13), we find

$$S^{ij}_B=S^{ij}+\frac{P^iP^kS^{jk}-P^jP^kS^{ik}}{\sqrt{2}P^-((P^2)^{1/2}+P^0)}+\frac{(P^iS^{(d-1)j}-P^jS^{(d-1)i})((P^2)^{1/2}+\sqrt{2}P^-)}{\sqrt{2}P^-((P^2)^{1/2}+P^0)}\tag{3.43a}$$

$$S^{(d-1)k}_B=S^{(d-1)k}-\frac{(P^2\delta^{ik}-P^iP^k)S^{(d-1)i}}{\sqrt{2}P^-((P^2)^{1/2}+P^0)}+\frac{((P^2)^{1/2}+\sqrt{2}P^-)P^iS^{ki}}{\sqrt{2}P^-((P^2)^{1/2}+P^0)},\tag{3.43b}$$

and so, since S^{ij}_B and $S^{(d-1)k}_B$ commute with D_B, we see from equation (3.41) that the effect of S^{ij}_B and $S^{(d-1)k}_B$ on the on-mass-shell states $|o,s,t;(p^-/\sqrt{2},\mathbf{0},-p^-/\sqrt{2}),u,\sigma]$ is exactly the same as the effect of S^{ij} and $S^{(d-1)k}$ on the off-mass-shell states $|o,s;(p^0,\mathbf{0},p^{(d-1)}),t,u,\sigma\rangle$.

3.4. Parity and time-reversal

We first give the effect of parity and time-reversal on the *off-mass-shell states*:

$$\mathscr{P}|o,s;p,t,u,\sigma\rangle=\varepsilon_P(t,u,\sigma)|o,s;Ip,t,u,\sigma\rangle,\tag{3.44a}$$

$$\mathscr{T}|o,s;p,t,u,\sigma\rangle=\varepsilon_T(t,u,\sigma)|o,s;Ip,t,u,-\sigma\rangle,\tag{3.44b}$$

where $I^\mu{}_\nu$ is the matrix with non-zero components $I^0{}_0=1$, $I^i{}_j=-\delta^i{}_j$, $I^{(d-1)}{}_{(d-1)}=-1$. The phase factors $\varepsilon_P(t,u,\sigma)$ and $\varepsilon_T(t,u,\sigma)$ are independent of p, though not necessarily t, u and σ, because $I(\lambda^{-1}Lp)=\lambda^{-1}L'(Ip)$ and $U(\lambda,L',0)=\mathscr{P}U(\lambda,L,0)\mathscr{P}^{-1}=\mathscr{T}U(\lambda,L,0)\mathscr{T}^{-1}$.

To find the effect of $\mathscr{P}$ and $\mathscr{T}$ on our *standard on-mass-shell state* $|o,s,t;(\kappa,\mathbf{0},\kappa),u,\sigma\}$, we use the definitions equations (3.19) and (3.41), together with equations (2.21), to give

$$\mathscr{P}|o,s,t;(\kappa,\mathbf{0},\kappa),u,\sigma\}=\varepsilon_P(t,u,\sigma)|o,s,t;(\kappa,\mathbf{0},-\kappa),u,\sigma],\tag{3.45a}$$

$$\mathscr{T}|o,s,t;(\kappa,\mathbf{0},\kappa),u,\sigma\}=\varepsilon_T(t,u,\sigma)|o,s,t;(\kappa,\mathbf{0},-\kappa),u,-\sigma],\tag{3.45b}$$

and for our *general on-mass-shell states*, defined by equation (3.23) and by

$$|o, s, t; p, u, \sigma] = \exp(-\mathrm{i}\alpha\hat{p}^i M^{(d-1)i})(p^0/\kappa)^{-\mathrm{i}M^{0(d-1)}}|o, s, t; (\kappa, \mathbf{0}, -\kappa), u, \sigma], \tag{3.46}$$

where $\tan\alpha = |\boldsymbol{p}|/p^{(d-1)}$, as before. We then find

$$\mathscr{P}|o, s, t; p, u, \sigma\} = \varepsilon_{\mathrm{P}}(t, u, \sigma)|o, s, t; Ip, u, \sigma], \tag{3.47a}$$

$$\mathscr{T}|o, s, t; p, u, \sigma\} = \varepsilon_{\mathrm{T}}(t, u, \sigma)|o, s, t; Ip, u, -\sigma]. \tag{3.47b}$$

We see that $\mathscr{P}$ and $\mathscr{T}$ transform one on-mass-shell Hilbert space into the other. Equations (3.45) and (3.47) also agree with equations (2.20) for the spin operators.

4. Off-mass-shell massless particles ($d = 4$)

We concentrate on those aspects particular to $d = 4$.

4.1. The light-cone transformations: effect on operators

4.1.1. Spin pseudovector and helicity operator. We define the Pauli–Lubanski spin pseudovector operator† $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}W_{\nu\rho}P_\sigma$ which, in terms of the spin operator $S^1 = S^{23}$, $S^2 = S^{31}$, $S^3 = S^{12}$ and momentum operator P^μ, is (Almond 1973a, equation (III.28))

$$W^\mu = \left(\boldsymbol{P}\cdot\boldsymbol{S} + P^3S^3, (P^2)^{1/2}\boldsymbol{S} + \frac{\boldsymbol{P}(\boldsymbol{P}\cdot\boldsymbol{S}+P^3S^3)}{(P^2)^{1/2}+P^0}, (P^2)^{1/2}S^3 + \frac{P^3(\boldsymbol{P}\cdot\boldsymbol{S}+P^3S^3)}{(P^2)^{1/2}+P^0}\right), \tag{4.1}$$

and the helicity operator of the off-mass-shell particle is

$$\Lambda = W^0/(\boldsymbol{P}^2 + (P^3)^2)^{1/2}. \tag{4.2}$$

Under the light-cone transformation generated by $D + M^{03}$, we find, using equations (3.2) on $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}W_{\nu\rho}P_\sigma$,

$$\lim_{\alpha\to\infty} \mathrm{e}^{-\mathrm{i}\alpha(D+M^{03})}W^\mu\,\mathrm{e}^{\mathrm{i}\alpha(D+M^{03})} = S_A^{12}(P^+/\sqrt{2}, \mathbf{0}, P^+/\sqrt{2}), \tag{4.3a}$$

$$\lim_{\alpha\to\infty} \mathrm{e}^{-\mathrm{i}\alpha(D+M^{03})}\Lambda\,\mathrm{e}^{\mathrm{i}\alpha(D+M^{03})} = S_A^{12}, \tag{4.3b}$$

and, under the light-cone transformation generated by $D - M^{03}$, we find, using equations (3.36) on $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}W_{\nu\rho}P_\sigma$,

$$\lim_{\alpha\to\infty} \mathrm{e}^{-\mathrm{i}\alpha(D-M^{03})}W^\mu\,\mathrm{e}^{\mathrm{i}\alpha(D-M^{03})} = -S_B^{12}(P^-/\sqrt{2}, \mathbf{0}, -P^-/\sqrt{2}), \tag{4.4a}$$

$$\lim_{\alpha\to\infty} \mathrm{e}^{-\mathrm{i}\alpha(D-M^{03})}\Lambda\,\mathrm{e}^{\mathrm{i}\alpha(D-M^{03})} = -S_B^{12}. \tag{4.4b}$$

4.1.2. Canonical form of generators. In equations (3.14) for $d = 4$, let us write $J^1 = -M^{23}$, $J^2 = -M^{31}$, $J^3 = -M^{12}$, $N^1 = -M^{01}$, $N^2 = -M^{02}$, $N^3 = -M^{03}$ together with

† Our convention is $\varepsilon^{0123} = +1$.

$S_A^3(p) \equiv S_A^{12}(p)$. The canonical form then becomes

$$
\begin{aligned}
&J^1 \to -\mathrm{i}\left(p^2\frac{\partial}{\partial p^3} - p^3\frac{\partial}{\partial p^2}\right) + \frac{p^1 S_A^3(p)}{p^0+p^3}, \qquad J^2 \to -\mathrm{i}\left(p^3\frac{\partial}{\partial p^1} - p^1\frac{\partial}{\partial p^3}\right) + \frac{p^2 S_A^3(p)}{p^0+p^3},\\
&J^3 \to -\mathrm{i}\left(p^1\frac{\partial}{\partial p^2} - p^2\frac{\partial}{\partial p^1}\right) + S_A^3(p), \qquad N^1 \to -\mathrm{i}p^0\frac{\partial}{\partial p^1} - \frac{p^2 S_A^3(p)}{p^0+p^3},\\
&N^2 \to -\mathrm{i}p^0\frac{\partial}{\partial p^2} + \frac{p^1 S_A^3(p)}{p^0+p^3}, \qquad N^3 \to -\mathrm{i}p^0\frac{\partial}{\partial p^3},\\
&D \to -\mathrm{i}\left(p^1\frac{\partial}{\partial p^1} + p^2\frac{\partial}{\partial p^2} + p^3\frac{\partial}{\partial p^3} + 1\right),
\end{aligned}
\tag{4.5}
$$

where the derivatives are taken with $p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = \text{constant} = 0$. The forms for the Lorentz transformation generators are those found previously by Lomont and Moses (1962) and Chakrabarti (1966) for massless irreducible representations of the Poincaré group.

4.2. *The light-cone transformation: effect on states*

We shall now consider the effect of the operator $e^{i\alpha(D+M^{03})}$ with $\alpha \to \infty$ on a general off-mass-shell state $|o, s; p, \sigma\rangle$ of spin s, four-momentum p^μ and S^3-eigenvalue σ (which runs from $+s$ to $-s$)†. The properties of such states are exactly the same as those of states with non-zero on-mass-shell mass (see Almond 1973a, § III.1.C), and the effect of a unitary operator of the Weyl group on such a state is (Almond 1973a, equation (III.64))

$$U(\lambda, L, a)|o, s; p, \sigma\rangle = \lambda^{-2} e^{ip'\cdot a} \sum_{\sigma'=-s}^{s} |o, s; p, \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(p)), \tag{4.6}$$

where $p' = \lambda^{-1}Lp$, $L^{-1}(p)$ is defined in equation (2.11), and $D^s_{\sigma'\sigma}(R)$ is the unitary irreducible representation of SO(3). So for finite α we have

$$e^{i\alpha(D+M^{03})}|o, s; p, \sigma\rangle = e^{-2\alpha} \sum_{\sigma'=-s}^{s} |o, s; p', \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(p)), \tag{4.7}$$

where $p' = e^{-\alpha}Lp$ and

$$L^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}. \tag{4.8}$$

We evaluate the Wigner rotation $L^{-1}(p')LL(p)$ for finite α and then let $\alpha \to \infty$, and find

$$(\lim_{\alpha\to\infty} L^{-1}(p')LL(p))^\mu{}_\nu = \begin{pmatrix} 1 & 0 \quad 0 & 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \delta^k{}_j + \dfrac{p^k p_j}{((p^2)^{1/2}+p^0)\sqrt{2}p^+} & -\dfrac{p^k((p^2)^{1/2}+\sqrt{2}p^+)}{((p^2)^{1/2}+p^0)\sqrt{2}p^+} \\ 0 & -\dfrac{p_j((p^2)^{1/2}+\sqrt{2}p^+)}{((p^2)^{1/2}+p^0)\sqrt{2}p^+} & 1 - \dfrac{p^2}{((p^2)^{1/2}+p^0)\sqrt{2}p^+} \end{pmatrix}, \tag{4.9}$$

† We emphasise again that the entire analysis is also valid for the light-cone limit of the *massive* particle states $|m, s; p, \sigma\rangle$.

which we shall call R_∞. So, we find for the on-mass-shell limit of $|o, s; p, \sigma\rangle$

$$\frac{1}{\sqrt{\delta(0)}} \lim_{\alpha\to\infty} e^{2\alpha} e^{i\alpha(D+M^{03})}|o, s; p, \sigma\rangle = \sum_{\sigma'=-s}^{s} |o, s; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), \sigma'\} D^s_{\sigma'\sigma}(R_\infty), \tag{4.10}$$

i.e. the mass-shell limit of the general off-mass-shell state $|o, s; p, \sigma\rangle$ is a superposition of the $|o, s; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), \sigma'\}$, except, of course, when $p^\mu = (p^0, \mathbf{0}, p^3)$, when we get $|o, s; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), \sigma\}$. The σ in $|o, s; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), \sigma\}$, being the eigenvalue of S_A^{12}, is also the helicity of the particle (see equation (4.3*b*)).

For the states $|o, s; (p^-/\sqrt{2}, \mathbf{0}, -p^-/\sqrt{2}), \sigma]$ defined by

$$|o, s; (p^-/\sqrt{2}, \mathbf{0}, -p^-/\sqrt{2}), \sigma] = (1/\sqrt{\delta(0)}) \lim_{\alpha\to\infty} e^{2\alpha} e^{i\alpha(D-M^{03})}|o, s; (p^0, \mathbf{0}, p^3), \sigma\rangle, \tag{4.11}$$

σ, being the eigenvalue of S_B^{12}, is *minus* the helicity of the particle (see equation (4.4*b*)).

The general on-mass-shell state is defined by

$$|o, s; p, \sigma\} = U(1, \mathscr{L}(p), 0)|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\},$$

where $\mathscr{L}(p)$ is the product of a boost in the 3-direction and a rotation which takes the 3-axis into the $(\boldsymbol{p}, p^3)$-direction (see equation (3.24)). Note that σ is unchanged by any Weyl group operator, i.e. the helicity is an invariant, as we expect. (In fact, we could denote the states $|o, s, t; p\}$ with $t=\sigma$ the helicity, though the quoted form is more conventional.) Under a unitary operator of the Weyl group, the state $|o, s; p, \sigma\}$ transforms as

$$U(\lambda, L, a)|o, s; p, \sigma\} = \lambda^{-1} e^{ia\cdot p'}|o, s; p, \sigma\} e^{-i\alpha_{12}\sigma} \tag{4.12}$$

(cf equation (3.35)), where $p' = \lambda^{-1}Lp$ and α_{12} is a function of $\mathscr{L}^{-1}(Lp)L\mathscr{L}(p)$.

The effect of parity and time-reversal on the off-mass-shell states $|o, s; p, \sigma\rangle$ is

$$\mathscr{P}|o, s; p, \sigma\rangle = \eta_P^*|o, s; Ip, \sigma\rangle, \tag{4.13a}$$

$$\mathscr{T}|o, s; p, \sigma\rangle = \eta_T^*(-)^{j+\sigma}|o, s; Ip, -\sigma\rangle, \tag{4.13b}$$

where our notation is that of Weinberg (1964b, § IV). So, using the definitions of our standard on-mass-shell states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ and $|o, s; (\kappa, \mathbf{0}, -\kappa), \sigma]$, i.e. equations (3.19) and (3.41) with $d=4$, we find

$$\mathscr{P}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\} = \eta_P^*|o, s; (\kappa, \mathbf{0}, -\kappa), \sigma], \tag{4.14a}$$

$$\mathscr{T}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\} = \eta_T^*(-)^{j+\sigma}|o, s; (\kappa, \mathbf{0}, -\kappa), -\sigma], \tag{4.14b}$$

and for the general on-mass-shell states of equations (3.23) and (3.46), we find

$$\mathscr{P}|o, s; p, \sigma\} = \eta_P^*|o, s; Ip, \sigma], \tag{4.15a}$$

$$\mathscr{T}|o, s; p, \sigma\} = \eta_T^*(-)^{j+\sigma}|o, s; Ip, -\sigma]. \tag{4.15b}$$

The significance of the two Hilbert spaces may be appreciated by considering an off-mass-shell neutrino of helicity $-\frac{1}{2}$, moving in the positive 3-direction. Such a state is denoted $|\nu; (p^0, \mathbf{0}, p^3), \sigma=-\frac{1}{2}\rangle$ (with $p^3>0$). If we put this onto the mass-shell using $D+M^{03}$, we obtain the state $|\nu; (p^+/\sqrt{2}, \mathbf{0}, p^+/\sqrt{2}), \sigma=-\frac{1}{2}\}$, i.e. an on-mass-shell neutrino of negative helicity. But if we put it onto the mass-shell using $D-M^{03}$, we obtain $|\nu; (p^-/\sqrt{2}, \mathbf{0}, -p^-/\sqrt{2}), \sigma=-\frac{1}{2}]$, i.e. an on-mass-shell neutrino of positive helicity! Quite generally, we must work in *one* of the Hilbert spaces (in the example of the neutrino given above it is $|\sigma=-\frac{1}{2}\}$) and then implement parity and time-reversal (if applicable) in the usual way (see e.g. Weinberg 1964a, § IX) by identifying $\mathscr{P}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ and $\mathscr{T}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ with the states $U(1, R_c, 0)|o, s; (\kappa, \mathbf{0}, \kappa), -\sigma\}$ and $U(1, R_c, 0)|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ respectively (both up to a phase factor), where R_c is a fixed rotation that takes $(\kappa, \mathbf{0}, \kappa)$ into $(\kappa, \mathbf{0}, -\kappa)$.

4.3. *Weinberg's theorem*

Weinberg (1964a) has shown that massless irreducible representations of the Poincaré group can be described by only certain irreducible representations of the homogeneous Lorentz group (see also Niederer and O'Raifeartaigh 1974, ch VII). We outline the proof below.

Let us define†

$$K^i=\tfrac{1}{2}(J^i+\mathrm{i}N^i), \qquad L^i=\tfrac{1}{2}(J^i-\mathrm{i}N^i) \qquad (i=1,2,3), \tag{4.16}$$

where $J^i=-\frac{1}{2}\varepsilon^{ijk}M^{jk}$ and $N^i=-M^{0i}$. These operators satisfy the Lie algebra of $\mathrm{SO}(3)\otimes\mathrm{SO}(3)$,

$$[K^i, K^j]=\mathrm{i}\varepsilon^{ijk}K^k, \qquad [L^i, L^j]=\mathrm{i}\varepsilon^{ijk}L^k, \qquad [K^i, L^j]=0, \tag{4.17}$$

so that irreducible representations of this algebra are labelled by $\boldsymbol{K}^2=k(k+1)$ and $\boldsymbol{L}^2=l(l+1)$, and states within an irreducible representation are labelled by the eigenvalues of K^3 and L^3. Hence an on-mass-shell standard state of a massless particle $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ can be described by the states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}^{(k,l)}_{(k^3,l^3)}$, which transform as the (k, l) irreducible representation of the homogeneous Lorentz group. We can rewrite equations (4.16) as

$$M^{12}=-(K^3+L^3), \tag{4.18a}$$

$$M^{03}=\mathrm{i}(K^3-L^3), \tag{4.18b}$$

$$-(M^{-2}+\mathrm{i}M^{-1})/\sqrt{2}=K^1-\mathrm{i}K^2, \tag{4.18c}$$

$$-(M^{-2}-\mathrm{i}M^{-1})/\sqrt{2}=L^1+\mathrm{i}L^2, \tag{4.18d}$$

$$(M^{+2}-\mathrm{i}M^{+1})/\sqrt{2}=K^1+\mathrm{i}K^2, \tag{4.18e}$$

$$(M^{+2}+\mathrm{i}M^{+1})/\sqrt{2}=L^1-\mathrm{i}L^2. \tag{4.18f}$$

We recall (equations (3.31*a*, *b*))

$$\begin{aligned} &M^{-1}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}=0=M^{-2}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}, \\ &M^{12}|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}=-\sigma|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}, \end{aligned} \tag{4.19}$$

so equations (4.18*c*) and (4.18*d*) tell us that

$$\begin{aligned} &(K^1-\mathrm{i}K^2)|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}=0, \\ &(L^1+\mathrm{i}L^2)|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}=0, \\ &(K^3+L^3)|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}=\sigma|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}, \end{aligned} \tag{4.20}$$

so the only states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}^{(k,l)}_{(k^3,l^3)}$ which are allowed are those with $k^3=-k$, $l^3=l$,

† In § 4.3 we drop the convention that latin italic indices run only from 1 to 2.

$(k^3+l^3)=\sigma$, or, in other words, the $k^3=-k$, $l^3=l$ component of the (k, l) irreducible representation, where

$$(k, l)=(k, k+\sigma), \qquad k=0, \tfrac{1}{2}, 1, \ldots \quad \text{for } \sigma>0, \tag{4.21a}$$

$$(k, l)=(l+|\sigma|, l), \qquad l=0, \tfrac{1}{2}, 1, \ldots \quad \text{for } \sigma<0, \tag{4.21b}$$

which is the result of Weinberg.

We now ask whether we can say anything about the states defined by equations (4.21) *for different* σ, i.e. we consider the $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ as the on-mass-shell limit of the off-mass-shell states $|o, s; (p^0, \mathbf{0}, p^3), \sigma\rangle$ with $\sigma=-s \ldots s$. (Remember that we could apply the same considerations to the light-cone limit states $|m, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ of a massive particle.) We recall that, in the off-shell case, the states of different σ (i.e. different eigenvalues of S^{12}) are related by the raising and lowering operators $-S^{32}\pm \mathrm{i}S^{31}$. When we go onto the light-cone, these become the unphysical operators $-S_A^{32}\pm \mathrm{i}S_A^{31}$ (see discussion after equations (3.9)). However, $-S_A^{32}$, S_A^{31} and S_A^{12} still satisfy the Lie algebra of SO(3) (see equations (3.6)), and therefore $-S_A^{32}\pm \mathrm{i}S_A^{31}$ are the raising and lowering operators for the states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ (or $|m, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$). But from equations (3.6) and (3.8), we see that

$$[-S_A^{32}\pm \mathrm{i}S_A^{31}, M^{03}]=0, \tag{4.22}$$

so that the states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}^{(k,l)}_{(-k,l)}$ (or $|m, s; (\kappa, \mathbf{0}, \kappa), \sigma\}^{(k,l)}_{(-k,l)}$) with different values of σ *all have the same eigenvalue of* M^{03}, which, by equation (4.18*b*), is just $-\mathrm{i}(k+l)$. Since, from equation (3.31*c*), the eigenvalue of $D+M^{03}$ for these states is i (i.e. $\mathrm{i}\frac{1}{2}(d-2)$ with $d=4$), we see that the eigenvalue of D is $\mathrm{i}(1+k+l)$.

So we now have our final result which can be stated thus: the description of the state $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$ (or $|m, s; (\kappa, \mathbf{0}, \kappa), \sigma\}$) of a spin multiplet by the (k, l) (with $-k+l=\sigma$) representation of the homogeneous Lorentz group automatically fixes the description of the other states $|o, s; (\kappa, \mathbf{0}, \kappa), \sigma+n\}$ (or $|m, s; (\kappa, \mathbf{0}, \kappa), \sigma+n\}$) ($n$ is a positive or negative integer such that $|\sigma+n|\leq s$) to be by the $(k-\frac{1}{2}n, l+\frac{1}{2}n)$ representation of the homogeneous Lorentz group. This means that, for even spin s, the lowest-order representation that describes $\sigma=0$ is $(\frac{1}{2}s, \frac{1}{2}s)$, and, for odd spin s, the lowest-order representation that describes $\sigma=\frac{1}{2}$ is $(\frac{1}{2}(s-\frac{1}{2}), \frac{1}{2}(s+\frac{1}{2}))$. These ideas can be translated into the language of quantum field theory when we take the $|\boldsymbol{p}|/m\to\infty$ limit of massive fields. For example, our theorem tells us that the lowest-order sets of fields that describe the massless limit of a massive spin-one particle are

$$\begin{array}{llll} \sigma=+1 & (0, 1) & & (\frac{1}{2}, \frac{3}{2}) \\ \sigma=0 & (\frac{1}{2}, \frac{1}{2}) & \text{and} & (1, 1) \\ \sigma=-1 & (1, 0) & & (\frac{3}{2}, \frac{1}{2}), \end{array} \tag{4.23}$$

where, in terms of the massive spin-one field $V^\mu(x)$ satisfying

$$(\Box+m^2)V^\mu=0, \qquad \partial^\mu V_\mu=0, \tag{4.24}$$

the spin-one components of the various fields can be written as

$$\begin{array}{ll} (\frac{1}{2}, \frac{1}{2}): & A^\mu=V^\mu, \\ (0, 1)\oplus(1, 0): & F^{\mu\nu}=(\partial^\nu V^\mu-\partial^\mu V^\nu)/m, \\ (1, 1): & A^{\mu\nu}=(\partial^\mu V^\nu+\partial^\nu V^\mu)/m, \\ (\frac{1}{2}, \frac{3}{2})\oplus(\frac{3}{2}, \frac{1}{2}): & T^{\mu\nu\rho}=\partial^\rho(\partial^\mu V^\nu-\partial^\nu V^\mu)/m^2+\frac{1}{3}(g^{\mu\rho}V^\nu-g^{\nu\rho}V^\mu), \end{array} \tag{4.25}$$

i.e. (0, 1) and (1, 0) are the chiral components of an antisymmetric second-rank tensor, (1, 1) is a traceless symmetric second-rank tensor, and $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{3}{2}, \frac{1}{2})$ are the chiral components of a third-rank tensor antisymmetric in two indices and traceless in all pairs of indices. On expanding V^μ in terms of helicity plane-wave functions

$$\begin{aligned}
&V^\mu(x) = \sum_{\lambda=0,\pm1} \int d^3p\, (e^\mu(\boldsymbol{p}, \lambda) a(\boldsymbol{p}, \lambda) f_{\boldsymbol{p}}(x) + e^\mu(\boldsymbol{p}, \lambda)^* a^\dagger(\boldsymbol{p}, \lambda) f_{\boldsymbol{p}}^*(x)),\\
&f_{\boldsymbol{p}}(x) = (1/\sqrt{(2\pi)^3 E(\boldsymbol{p})})\, e^{-ip\cdot x},\\
&e^\mu(\boldsymbol{p}, +1) = -(0, (\boldsymbol{\varepsilon}(\boldsymbol{p}, 1) + i\boldsymbol{\varepsilon}(\boldsymbol{p}, 2))/\sqrt{2}),\\
&e^\mu(\boldsymbol{p}, 0) = (|\boldsymbol{p}|/m, \boldsymbol{p}E(\boldsymbol{p})/|\boldsymbol{p}|m),\\
&e^\mu(\boldsymbol{p}, -1) = (0, (\boldsymbol{\varepsilon}(\boldsymbol{p}, 1) - i\boldsymbol{\varepsilon}(\boldsymbol{p}, 2))/\sqrt{2}),
\end{aligned} \tag{4.26}$$

where $\boldsymbol{p}$, $\boldsymbol{\varepsilon}(\boldsymbol{p}, 1)$ and $\boldsymbol{\varepsilon}(\boldsymbol{p}, 2)$ form a mutually perpendicular set. We easily see that, in the $|\boldsymbol{p}|/m \to \infty$ limit, only the required helicity components survive.

5. Conclusion

By using the infinite transformations generated by D_A and D_B we have been able to take off-mass-shell massless particles onto the mass-shell, and have been able to describe the particles by using the transformed operators.

One point which has so far failed to emerge from this work, or our previous one on massive particles (Almond 1973a, § III), is the connection between the space–time position operator R^μ and the localised off-mass-shell states given by (Almond 1973a, equation (III.65))

$$|m, s; r, \sigma) = \frac{1}{(2\pi)^2} \int d^4p\, \theta(p^2)\theta(p^0)\, e^{ip\cdot r} |m, s; p, \sigma\rangle \tag{5.1}$$

(for $d = 4$), and also valid for $m = 0$. The elucidation of this problem is an important goal for future work.

Appendix 1. The group of inhomogeneous Galilei transformations and the dilatation $(t', \boldsymbol{x}') = (\lambda^2 t, \lambda \boldsymbol{x})$ in Δ space dimensions

This is a $[\frac{1}{2}(\Delta+1)(\Delta+2)+1]$-parameter group consisting of space rotations, pure Galilei transformations, displacements in space and time, and the non-relativistic dilatations acting on a Δ-dimensional Euclidean space plus time:

$$\boldsymbol{x}' = \lambda R\boldsymbol{x} + \lambda^2 \boldsymbol{v}t + \boldsymbol{a}, \qquad t' = \lambda^2 t + b, \tag{A1.1}$$

where R is a rotation matrix, $\boldsymbol{v}$ is a boost velocity, $\boldsymbol{a}$ is a displacement in space, b is a displacement in time, and λ is a dilatation. It has been studied in detail for $\Delta = 3$ (Almond 1973a, § II, 1974, Bez 1976), and it has been shown (Almond 1973a, appendix A) that, because in quantum mechanics we are looking for unitary *ray* representations† of the group, i.e. unitary operators $U(G)$ satisfying

$$U(G')U(G) = e^{i\xi(G', G)} U(G'G), \tag{A1.2}$$

† The classic paper on the subject is that of Bargmann (1954). There are several reviews available (Hamermesh 1962, ch 12, Lévy-Leblond 1971, § III.A, Almond 1973b).

with G', G two group elements and $\xi(G', G)$ a real phase factor, the structure we are studying in Hilbert space is not the Lie algebra of the group, but a certain central extension of it. The same central extension occurs for general Δ, and it is given by

$$\begin{aligned}
&[J^{ij}, J^{kl}] = -\mathrm{i}(J^{ik}\delta^{jl} - J^{il}\delta^{jk} + J^{jl}\delta^{ik} - J^{jk}\delta^{il}), \qquad [J^{ij}, K^k] = \mathrm{i}(K^i\delta^{jk} - K^j\delta^{ik}),\\
&[J^{ij}, P^k] = \mathrm{i}(P^i\delta^{jk} - P^j\delta^{ik}), \qquad [K^i, H] = -\mathrm{i}P^i, \qquad [K^i, P^j] = -\mathrm{i}M\delta^{ij},\\
&[J^{ij}, H] = [K^i, K^j] = [P^i, P^j] = [P^i, H] = 0, \\
&[D, J^{ij}] = 0, \qquad [D, K^i] = \mathrm{i}K^i, \qquad [D, P^i] = -\mathrm{i}P^i,\\
&[D, H] = -2\mathrm{i}H, \qquad [J^{ij}, M] = [K^i, M] = [P^i, M] = [H, M] = [D, M] = 0,
\end{aligned} \tag{A1.3}$$

where J^{ij}, K^i, P^i, H and D are the Hermitian Hilbert space generators of rotations, boosts, space displacements, time displacements and dilatations, respectively, and M is the Hermitian Hilbert space generator of the central extension.

Let us now define the spin, time and position operators by

$$S^{ij} = -J^{ij} + (K^iP^j - K^jP^i)/M, \tag{A1.4a}$$

$$T = \frac{1}{4}\left[\frac{1}{U}, D\right]_+ - \frac{[K^i, P^i]_+}{4MU}, \tag{A1.4b}$$

$$R^i = \frac{TP^i - K^i}{M} = \frac{1}{4M}\left[\frac{P^i}{U}, D\right]_+ - \frac{[K^j, P^jP^i]_+}{4M^2U} - \frac{K^i}{M}, \tag{A1.4c}$$

respectively, where we have written $U \equiv (H - \boldsymbol{P}^2/2M)$ for the internal energy operator. These operators have the following commutation relations with the generators,

$$\begin{aligned}
&[J^{ij}, S^{kl}] = -\mathrm{i}(S^{ik}\delta^{jl} - S^{il}\delta^{jk} + S^{jl}\delta^{ik} - S^{jk}\delta^{il}),\\
&[K^i, S^{kl}] = [P^i, S^{kl}] = [H, S^{kl}] = [D, S^{kl}] = [M, S^{kl}] = 0,\\
&[J^{ij}, T] = 0, \qquad [K^i, T] = 0, \qquad [P^i, T] = 0,\\
&[H, T] = \mathrm{i}, \qquad [D, T] = 2\mathrm{i}T, \qquad [M, T] = 0,\\
&[J^{ij}, R^k] = \mathrm{i}(R^i\delta^{jk} - R^j\delta^{ik}), \qquad [K^i, R^j] = -\mathrm{i}\delta^{ij}T,\\
&[P^i, R^j] = -\mathrm{i}\delta^{ij}, \qquad [D, R^i] = \mathrm{i}R^i, \qquad [M, R^i] = 0,
\end{aligned} \tag{A1.5}$$

and with each other,

$$\begin{aligned}
&[S^{ij}, S^{kl}] = \mathrm{i}(S^{ik}\delta^{jl} - S^{il}\delta^{jk} + S^{jl}\delta^{ik} - S^{jk}\delta^{il}),\\
&[S^{ij}, T] = [S^{ij}, R^i] = [T, R^i] = [R^i, R^j] = 0,
\end{aligned} \tag{A1.6}$$

all as we should expect.

Equations (A1.5) and (A1.6) describe a virtual (off-energy-shell) non-relativistic particle, and the invariant operators of the algebra are M giving the mass, $\frac{1}{2}S^{ij}S^{ij}$ which for $\Delta \leq 3$ gives the spin of the particle (for $\Delta > 3$, there are other spin-type invariants, e.g. for $\Delta = 4$, $\frac{1}{4}\varepsilon^{ijkl}S^{ij}S^{kl}$ is also invariant), and sign (U). Since the internal energy U is arbitrary for a non-relativistic particle (Lévy-Leblond 1963, Almond 1973a, § II), a virtual non-relativistic particle is described group-theoretically by the direct sum of irreducible representations with the same mass and spin but with sign $(U) = \pm 1$, as shown by Bez (1976).

Appendix 2. Poisson brackets and Dirac brackets

The Poisson brackets of R^μ, P^μ and $W^{\mu\nu}$ are obtained from the commutators, equations (2.7), by the replacement $[A, B]/\mathrm{i} \to \{A, B\}$. We wish to construct Dirac brackets (Dirac 1958, 1964, Hanson *et al* 1976) compatible with the second-class constraints

$$\varphi_1 \equiv P^2 \approx 0, \tag{A2.1a}$$

$$\varphi_2 \equiv R^+ - \beta \approx 0. \tag{A2.1b}$$

We can construct from any dynamical variable A, a new variable A':

$$A \to A' = A - \{A, \varphi_\alpha\} C^{-1}_{\alpha\beta} \varphi_\beta, \tag{A2.2}$$

where

$$C_{\alpha\beta} = \{\varphi_\alpha, \varphi_\beta\} = 2P^+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C^{-1}_{\alpha\beta} = \frac{1}{2P^+} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A2.3}$$

This procedure is quite general, and normally we find that A' is weakly equal to A,

$$A' \approx A, \tag{A2.4}$$

and, taking the Poisson bracket of equation (A2.2) with a second-class constraint φ_γ,

$$\{A', \varphi_\gamma\} \approx 0. \tag{A2.5}$$

In other words, A' is a quantity which is compatible with the constraints. However, in this particular case, this is not so, since $\{A, R^+ - \beta\}$ is $\sim 1/P^2$ for $A = R^{\mu(\neq +)}$ or $W^{\mu\nu}$. In fact

$$\begin{aligned}
&R^+ \to \beta, \qquad R^- \to R^- + W^{-+}/2P^+ - P^-(R^+ - \beta)/P^+, \\
&R^i \to R^i - W^{+i}/2P^+ - P^i(R^+ - \beta)/P^+, \\
&P^+ \to P^+, \qquad P^- \to P^- - P^2/2P^+, \qquad P^i \to P^i, \\
&W^{+i} \to \tfrac{3}{2} W^{+i}, \qquad W^{-i} \to W^{-i} + (P^- W^{+i} + P^i W^{-+})/2P^+, \\
&W^{ij} \to W^{ij} - (W^{+i}P^j - W^{+j}P^i)/2P^+, \qquad W^{-+} \to \tfrac{3}{2} W^{-+}.
\end{aligned} \tag{A2.6}$$

We therefore see that the quantities R^+, $R^- - W^{-+}/P^+$, $R^i + W^{+i}/P^+$, P^+, P^-, P^i, $-W^{ij} - (W^{+i}P^j - W^{+j}P^i)/P^+$ and $-(P^2)^{1/2} W^{+i}/P^+$ are weakly unchanged under equation (A2.2), and are therefore the relevant dynamical variables to study. But they are the same quantities which were found in § 3.1.1 and whose Poisson brackets with each other are given by equations (3.6) with $[A, B]/\mathrm{i} \to \{A, B\}$. Their Dirac brackets are given by the formula

$$\{A, B\}^* = \{A, B\} - \{A, \varphi_\alpha\} C^{-1}_{\alpha\beta} \{\varphi_\beta, B\}, \tag{A2.7}$$

and we find that the Dirac brackets are equal to the Poisson brackets except for

$$\left\{R^i + \frac{W^{+i}}{P^+}, P^-\right\}^* = \frac{P^i}{P^+}, \qquad \{R^+, P^-\}^* = 0, \qquad \left\{R^- - \frac{W^{-+}}{P^+}, P^-\right\}^* = \frac{P^-}{P^+}. \tag{A2.8}$$

Appendix 3. An operator identity

Consider three operators A, B, C satisfying the commutation relations

$$[A, B] = \mathrm{i}C, \qquad [C, A] = \mathrm{i}A, \qquad [C, B] = -\mathrm{i}B, \tag{A3.1}$$

i.e. the Lie algebra of SO(2, 1). (We are thinking in terms of $A = M^{+i}$, $B = M^{-i}$, $C = M^{0(d-1)}$, but our treatment is quite general.) Let us look for an expression for $\exp(\mathrm{i}\theta(A-B))$ of the form

$$\mathrm{e}^{\mathrm{i}\theta(A-B)} = \mathrm{e}^{\mathrm{i}h(\theta)A}\, \mathrm{e}^{\mathrm{i}f(\theta)C}\, \mathrm{e}^{\mathrm{i}g(\theta)B}. \tag{A3.2}$$

On differentiating with respect to θ, we find

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta(A-B)}\mathrm{i}(A-B) &= \mathrm{e}^{\mathrm{i}h(\theta)A}\mathrm{i}h'(\theta)A\, \mathrm{e}^{\mathrm{i}f(\theta)C}\, \mathrm{e}^{\mathrm{i}g(\theta)B} \\ &\quad + \mathrm{e}^{\mathrm{i}h(\theta)A}\, \mathrm{e}^{\mathrm{i}f(\theta)C}\, \mathrm{i}f'(\theta)C\, \mathrm{e}^{\mathrm{i}g(\theta)B} + \mathrm{e}^{\mathrm{i}h(\theta)A}\, \mathrm{e}^{\mathrm{i}f(\theta)C}\, \mathrm{e}^{\mathrm{i}g(\theta)B}\, \mathrm{i}g'(\theta)B.\end{aligned} \tag{A3.3}$$

We must now commute the operators A and C in the first two terms past the exponentials to the right of them by expanding the exponentials as power series and using the identity (valid for any two operators P and Q)

$$[P, Q^n] = \sum_{j=0}^{n-1} Q^j [P, Q] Q^{n-1-j}. \tag{A3.4}$$

We eventually find

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta(A-B)}\mathrm{i}(A-B) &= \mathrm{e}^{\mathrm{i}h(\theta)A}\, \mathrm{e}^{\mathrm{i}f(\theta)C}\, \mathrm{e}^{\mathrm{i}g(\theta)B}\mathrm{i}(h'(\theta)\, \mathrm{e}^{f(\theta)}(A - g(\theta)C - \tfrac{1}{2}g^2(\theta)B) \\ &\quad + f'(\theta)C + f'(\theta)g(\theta)B + g'(\theta)B).\end{aligned} \tag{A3.5}$$

On equating the coefficients of A, B and C on the two sides of this equation, we find

$$h'(\theta) = \mathrm{e}^{-f(\theta)}, \qquad f'(\theta) = g(\theta), \qquad 2g'(\theta) = -(g^2(\theta) + 2), \tag{A3.6}$$

which must be solved with the boundary conditions $h(0) = f(0) = g(0) = 0$. This gives us

$$\begin{aligned} g(\theta) &= -\sqrt{2}\tan(\theta/\sqrt{2} + n\pi), \qquad f(\theta) = \ln\cos^2(\theta/\sqrt{2} + n\pi), \\ h(\theta) &= \sqrt{2}\tan(\theta/\sqrt{2} + n\pi) \qquad (n = \text{integer}),\end{aligned} \tag{A3.7}$$

and, on choosing the branch $n = 0$, we obtain our final result:

$$\mathrm{e}^{\mathrm{i}\theta(A-B)} = \mathrm{e}^{\mathrm{i}\sqrt{2}\tan(\theta/\sqrt{2})A}\, \mathrm{e}^{\mathrm{i}\ln\cos^2(\theta/\sqrt{2})C}\, \mathrm{e}^{-\mathrm{i}\sqrt{2}\tan(\theta/\sqrt{2})B}. \tag{A3.8}$$

Similarly, we find

$$\mathrm{e}^{\mathrm{i}\theta(A+B)} = \mathrm{e}^{\mathrm{i}\sqrt{2}\tanh(\theta/\sqrt{2})A}\, \mathrm{e}^{\mathrm{i}\ln\cosh^2(\theta/\sqrt{2})C}\, \mathrm{e}^{\mathrm{i}\sqrt{2}\tanh(\theta/\sqrt{2})B}. \tag{A3.9}$$

The corresponding formulae with the positions of A and B interchanged can be found by the substitution $A \to B$, $B \to A$, $C \to -C$ in equations (A3.8) and (A3.9):

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta(A-B)} &= \mathrm{e}^{-\mathrm{i}\sqrt{2}\tan(\theta/\sqrt{2})B}\, \mathrm{e}^{-\mathrm{i}\ln\cos^2(\theta/\sqrt{2})C}\, \mathrm{e}^{\mathrm{i}\sqrt{2}\tan(\theta/\sqrt{2})A}, \\ \mathrm{e}^{\mathrm{i}\theta(A+B)} &= \mathrm{e}^{\mathrm{i}\sqrt{2}\tanh(\theta/\sqrt{2})B}\, \mathrm{e}^{-\mathrm{i}\ln\cosh^2(\theta/\sqrt{2})C}\, \mathrm{e}^{\mathrm{i}\sqrt{2}\tanh(\theta/\sqrt{2})A}\end{aligned} \tag{A3.10}$$

Appendix 4. A note on the finite-dimensional irreducible representations of SO($d-1,1$)

Firstly, we note that the finite-dimensional (and therefore non-unitary) irreducible representations of the Lie algebra of SO($d-1,1$) are given by the irreducible representations of SO(d) which are generated by the operators M^{ij}, $M^{(d-1)i}$, $\mathrm{i}M^{0i}$, and $\mathrm{i}M^{0(d-1)}$. SO(d) has rank $K+1$, where K is the rank of SO($d-2$). The irreducible representations are labelled by the invariants a_l ($l=1\ldots K+1$), and states within an irreducible representation are labelled by the eigenvalue m of $\mathrm{i}M^{0(d-1)}$, the eigenvalues m_k ($k=1\ldots K$) of K mutually commuting M^{ij}, and the eigenvalues b_j ($j=1\ldots\frac{1}{2}[\frac{1}{2}d(d-1)-3(K+1)]$) of the extra labelling operators. We can write the Lie algebra in 'Weyl form' with the Hermitian operators

$$\mathrm{i}M^{0(d-1)}, M^{12}, M^{34}, M^{56}, \ldots. \qquad \text{(A4.1)}$$

The operators

$$M^{-1}\pm \mathrm{i}M^{-2}, M^{-3}\pm \mathrm{i}M^{-4}, M^{-5}\pm \mathrm{i}M^{-6}, \ldots \qquad \text{(A4.2)}$$

are raising operators for $\mathrm{i}M^{0(d-1)}$ and lowering/raising operators for M^{12}, M^{34}, $M^{56}, \ldots$. The operators

$$M^{+1}\pm \mathrm{i}M^{+2}, M^{+3}\pm \mathrm{i}M^{+4}, M^{+5}\pm \mathrm{i}M^{+6}, \ldots \qquad \text{(A4.3)}$$

are lowering operators for $\mathrm{i}M^{0(d-1)}$ and lowering/raising operators for M^{12}, M^{34}, $M^{56}, \ldots$. For d odd, the last operators in (A4.2) and (A4.3) are $M^{-(d-2)}$ and $M^{+(d-2)}$, respectively, which have no effect on M^{12}, M^{34}, $M^{56}, \ldots$. There are also the lowering/raising operators for M^{12},

$$M^{32}\pm \mathrm{i}M^{31}, M^{42}\pm \mathrm{i}M^{41}, \ldots, \qquad \text{(A4.4)}$$

and also for M^{34}, $M^{56}, \ldots$.

A state with the greatest eigenvalue $m_{\max}$ of $\mathrm{i}M^{0(d-1)}$ satisfies

$$(M^{-1}\pm \mathrm{i}M^{-2})\Psi^{(a)}_{(m_{\max},m,b)}=0, \qquad (M^{-3}\pm \mathrm{i}M^{-4})\Psi^{(a)}_{(m_{\max},m,b)}=0, \qquad \text{etc,} \qquad \text{(A4.5)}$$

and, by the theory of finite-dimensional representations of Lie algebras, will also satisfy

$$(M^{+1}+\mathrm{i}M^{+2})^{p+1}\Psi^{(a)}_{(m_{\max},m,b)}=0, \qquad (M^{+1}-\mathrm{i}M^{+2})^{q+1}\Psi^{(a)}_{(m_{\max},m,b)}=0, \qquad \text{etc,} \qquad \text{(A4.6)}$$

with $p, q, \ldots$ integers and $(M^{+1}+\mathrm{i}M^{+2})^{p}\Psi^{(a)}_{(m_{\max},m,b)}\neq 0$, etc. From equations (A4.6) we can derive

$$(n^iM^{+i})^{(p+q+\ldots)+1}\Psi^{(a)}_{(m_{\max},m,b)}=0 \qquad \text{(A4.7)}$$

(with n^i an arbitrary transverse unit vector) by expressing n^iM^{+i} as a linear combination of $M^{+1}\pm \mathrm{i}M^{+2}$, etc, expanding in a binomial series, and using equations (A4.6). (The proof is straightforward for $(d-2)=2$, and can be proved for higher d by induction.) The equation

$$[(n^iM^{+i})^{h+1}, (n^iM^{-i})]=(n^iM^{+i})^h(h+1)(\mathrm{i}M^{0(d-1)}-\tfrac{1}{2}h) \qquad (h=\text{integer}) \qquad \text{(A4.8)}$$

follows from equations (2.17) and (A3.4), and, on putting $h=(p+q+\ldots)$ and using equations (A4.5) and (A4.7), we see that the eigenvalue $m_{\max}$ of $\mathrm{i}M^{0(d-1)}$ is equal to $\frac{1}{2}(p+q+\ldots)$.

Let us now return to equation (3.25). Instead of the state $|o, s, t; (\kappa, \mathbf{0}, \kappa)u, \sigma\}$ we consider the states $|o, s, t; (\kappa, \mathbf{0}, \kappa), u, \sigma\}^{(a)}_{(m_{\max},-\sigma,b)}$ which transform as the $(m_{\max}, -\sigma, b)$

component of the (a) irreducible representation of $SO(d-1, 1)$. Since, on each of these states, the second exponential becomes $(p^0 \cos^2(\frac{1}{2}\alpha)/\kappa)^{m_{max}}$, and the first exponential terminates at the $(p^i M^{+i}/p^+)^{2m_{max}}$ term by equation (A4.7), giving a factor $\tan^{2m_{max}}(\frac{1}{2}\alpha)$, we see that equation (3.25) is, in fact, non-singular as $\alpha \to \pi$.

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† *Erratum.* The $\rho_1, \lambda_1, \lambda_2, \lambda_3$ in equation (3.13) of this paper are arbitrary functions of P^2 and τ, not constants. The choice $\rho_1 = -c/((P^2)^{1/2} + mc)$ in equation (3.15) gives $H = -((P^2)^{1/2}c - mc^2)$ with $f(P^2, \tau) = (P^2)^{1/2}c\tau$.

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